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ABSTRACT

This is a manual for teachers using SMSG high school programed text materials in algebra. The commentary is organized into four parts. The first part contains a discussion of ways to use this programed text. The second and main part consists of a chapter by chapter commentary on the text. The third part is a listing of topics keyed to a list of supplementary references from the volumes of the "New Mathematical Library" (NML), which is a series of expository monographs produced by the School Mathematics Study Group and aimed at the level of maturity of the secondary school pupil. The fourth part contains suggested test items. (MP)

School Mathematics Study Group

Programed First Course in Algebra

Revised Form H

Unit 63

Programed First Course in Algebra

Revised Form H

Teacher's Commentary

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NOTE: The original printing of the student text appeared in four parts. The present printing appears in two parts. References in this commentary to Part 1 or Part 2 of the text are, therefore, to Part I of the present student text, and references to Parts 3 or 4 are to Part II.

Ways to Use This Programed Text

A careful examination of the text will reveal certain basic features:

1. Although much of the information to be imparted is broken down into small steps, the book also contains passages of straight text, where we wish the student to consider a concept in its entirety, or where we wish to motivate a transition to a new idea, or for some similar purpose.

2. The student is required to respond in some way at each step, except in passages of straight text. A special response sheet for each section is provided. Frequently, the required mode of response is to write a word or numeral which completes the sentence. In other cases, the student is asked to indicate a choice of responses by circling a letter. From time to time exercises appear for which the correct answers are supplied at the back of the book. These exercises provide special opportunity for the student to write up solutions in complete and acceptable form.

3. The student's response is immediately reinforced by checking against correct responses which are furnished to him. Since most of these--except where he is referred to the answer key--are of necessity close to the items to which they correspond, they are in shaded zones. In the preface to the text, we suggest to the student that he will find it helpful to cover the shaded zone with a piece of paper or a card while he makes his response.

Of course, the student will gain little by blindly copying responses. The teacher should encourage the attitude that by so doing the student "cheats" himself. On the other hand, learning is an active process, and there seems little ground for concern if the student occasionally verifies his response before writing it. In particular, students should not feel that working the program is an extended test situation, and teachers should help them to define goals in terms of ideas and skills mastered, rather than of the number of responses completed. We hope that students will become aware that they can often learn from mistakes.

The classroom uses of the book may vary according to the abilities and the mathematical background of the students, from a completely self-paced technique to a more conventional situation in which the book is used as a basic text, with opportunities for class discussion and individual help.

For superior students, self-pacing may be especially effective, since it would enable such students to move rapidly. This would be particularly true in the early chapters which include much material with which they may already be familiar.

For students of average ability, part of the working of the program may be done in class, with the teacher free to move about, giving individual help and guidance. With the rest of the work done outside of class, the group will be enabled to move at a uniform pace, with time available for class discussion at frequent intervals.

The programed format also makes feasible several flexible arrangements:

1. A student who is absent for several days can be expected to gain more from working through the program on his own than he could learn from an ordinary text without special help.

2. A student who enters the class late in the year should be able to catch up more easily, with less extra assistance, than would be possible with a conventional text.

3. The student who often drops behind and becomes hopelessly lost in a conventional class may be able to continue at his own pace in this text, and thus salvage something from the year's work.

4. For students who need more help with details of the work, "branches" have been built in to provide this aid. Also, detailed solutions to many problems will be found in the answer key section.

5. From time to time "skips" are provided in order that the more able student is not forced to waste time on what he has already mastered, or to suffer endless repetition of details which are needed by the less able.

6. Starred exercises have been included in many places. These contain more challenging questions, and provide interesting work for the superior student while the less able student is engaged in proceeding slowly, or in "branching" over minor details.

The approach, in this text as in the SMSG First Course in Algebra on which this is based, is spiral in form; that is, nearly every concept introduced appears repeatedly later. For this reason, the teacher who is using a text of this sort for the first time should realize that complete mastery of many topics is not to be insisted upon when they are first met.

Since changes in the arrangement of some topics and in some details of treatment have been made in this final revision, it is suggested that even those teachers who are familiar with the First Course in Algebra should work through the program with care. This procedure is, of course, imperative for the teacher who is teaching SMSG Algebra for the first time.

The program is intended to provide a full year's course for a superior student with a good background in such topics as those included in the SMSG texts for grades 7 and 8. For less able students and students without such background, it is necessary to select, from the wealth of material in the text, those topics which are basic to an adequate course in first year algebra. In general, however, we suggest that Parts 1 and 2 should be covered in the first half of the school year.

For the average student whose background is largely traditional arithmetic, the unstarred items in the following chapters form a minimal course:

Chapters 1-18

Chapter 20

Chapter 22

Where time permits, this may be enriched by adding other sections or chapters, and by assigning starred items.

Although the construction of suitable chapter tests is left to the skill and judgment of the teacher, at the end of this volume (page 90) are suggested test items for each chapter, which have been compiled as an aid in testing for the goals of the chapter.

COURSE OUTLINE

OVERVIEW

Since the program is self-contained, in the sense that the student should be able to follow the details of the mathematical development, this outline is rather brief. We hope that the outline will guide the teacher in understanding the ideas that underlie the mathematical presentation.

The general plan of the course is to build upon the student's experience with arithmetic. We assume that the student is familiar with the operations of addition and multiplication in arithmetic. The student has probably had exposure to using a number line. In Chapters 1-4 the student is led to extract from his experience the fundamental properties of addition and multiplication. These chapters also introduce the language of sets, the notion of variable, and the concept of order on the number line. Much of this material may be familiar to many students. It is hoped that these chapters may be covered quickly. In the early chapters the student will become acquainted with the mechanics of the program while dealing with material which is not entirely new to him.

Chapter 5 is of a special nature. It is designed to assist the student in developing skill in using the language and symbolism of algebra in relation to word problems.

Part 2 (Chapters 6-11) begins with the introduction of the negative real numbers. This introduction is made by adjoining the set of negative real numbers to the set of numbers of arithmetic. The number line is used to provide a geometrically intuitive model. Some students have had experience with negative numbers, but our use of "opposites" will probably be new. Addition and multiplication are then defined for the real numbers. We base these definitions on the student's ability to add and multiply in arithmetic, on the definition of "opposite" and on the desire to preserve the fundamental properties. Thus, we "extend" these operations from the set of numbers of arithmetic to the set of real numbers. Adopting the same point of view, we extend the order relation "is less than" to the real numbers and develop the basic properties of this relation.

At the end of Chapter 10, the development of the real number system is summarized. The fundamental properties of addition, multiplication, and order are listed. These are axioms for an ordered field, although we do not use this terminology. The Completeness Axiom, which is needed for a full definition of the real number system, is not presented. At this point, we also list the various theorems which have been discussed. Chapter 11 contains definitions of subtraction and division in terms of addition and multiplication, respectively. This completes the extension of the arithmetic of the non-negative numbers to the real numbers.

Part 3 begins with a chapter on the factorization of positive integers. Some of this material will be in the nature of review to most students. It is included at this point for several reasons. It is needed in working with fractions and polynomials; and we wish to point out, later on, the analogies between factoring integers and factoring polynomials. In addition, the material is interesting in its own right and it offers a change of pace for the student.

Chapters 13-17 deal largely with topics that are usually considered as "elementary algebra". These topics are fractions, exponents, radicals, and polynomials. As manipulative skills are developed, the student is expected to understand the reasons which justify the manipulations. These reasons, in turn, depend on the earlier material; that is, on the fundamental properties of the real number system. Part 3 concludes with the application of the technique of completing the square to the general quadratic polynomial, $ax^2 + bx + c$.

More advanced topics are found in Part 4, including rational expressions, equivalent equations and inequalities. The real number plane, graphs of linear equations and inequalities, systems of linear equations and inequalities, and graphs of quadratic polynomials are the topics of Chapters 18-23.

Chapter 24 provides an introduction to the important concept of function. Only real valued functions of a real variable are considered. The rate of progress through the program will vary from student to student, but we hope that the better students will be able to complete the material on functions.

There is available a book written specifically to explain the mathematical thinking behind SMSG First Course in Algebra. This is Haag, Studies in Mathematics, Vol. III, Structure of Elementary Algebra. (Yale University, 1961.) Since the programed material has been developed from First Course, the Haag volume is a valuable reference.

INTRODUCTION TO PART 1

(Chapters 1-5)

The point of view adopted by the authors of this programed text is that the student of a first course in algebra has had a great deal of experience with the techniques of arithmetic. He may or may not have had some introduction to the algebraic structure underlying these techniques. Therefore, we begin by reviewing the operations of addition and multiplication for the numbers of arithmetic (the non-negative real numbers), and stating the properties of these operations in algebraic language. No attempt is made to prove these properties; the generalizations are made from numerical examples which are familiar to the student.

The first four chapters may be thought of as the introduction or "prologue" to the course. In these chapters we introduce the language of algebra. The idea of "set" plays an important role throughout this course, but we do not attempt to develop formal concepts from set theory.

A great deal of use is made of the "number line". Here we give a geometric interpretation of the numbers under consideration. In this part we establish a one-to-one correspondence between the numbers of arithmetic and all points of the line to the right of the number 0, and the number 0 itself. The concept of order is derived from this correspondence. "Less than" is defined to mean "to the left of" on the number line.

Although the operations of subtraction and division are familiar to the student, we do not emphasize these as separate operations since we shall concentrate on the development of the real number field with the operations of addition and multiplication.

At the end of this introduction is a summary of the properties of the operations of addition and multiplication for the numbers of arithmetic.

The final chapter of Part 1 serves to strengthen the use of the mathematical language developed, by concentrating on the mathematical interpretation of verbal problems.

Chapter 1

SETS AND THE NUMBER LINE

Students who have studied SMSG Mathematics for Junior High School will have had some experience with sets and the number line. They may be able to go through this chapter rather quickly but they should at least try the review problems to be sure they understand the concepts.

1-1. Sets and Subsets

Although all of the first sets listed at the outset of the chapter are not examples of sets of numbers, we move quickly in the text to consideration of such sets. At this time we introduce the notation used for sets. We also define intersection of sets and union of sets since these are useful concepts throughout the course. Although the symbols $\cup$ and $\cap$ are not used extensively throughout this text, the notation is introduced since it is both useful and familiar.

We define and distinguish between the set of whole numbers and the set of counting numbers or natural numbers, but we find the set of whole numbers more useful for our purposes. With the discussion of real numbers in Chapter 6, the term integers will be introduced to designate whole numbers and their opposites (negatives).

There are three common errors made by students in working with the empty set. The most common is the confusion of $\{0\}$ and $\emptyset$. Another mistake is to use the words "an empty set" or "a null set" instead of "the empty set". There is but one empty set, though it has many descriptions. A third error is the use of the symbol $\{\emptyset\}$ instead of $\emptyset$.

1-2. The Number Line

The number line is used as an illustrative and motivational device, and our discussion of it is quite intuitive and informal. As was the case with the preceding section, more questions are raised than can be answered immediately.

Present on the number line implicitly are points corresponding to the negative numbers, as is suggested by the reference to a thermometer scale and by the presence in the illustrations of the left side of the number line. Since, however, the plan of the course is to move directly to the consideration of the properties of the operations on the non-negative numbers, anything more than casual recognition of the existence of the negative numbers at this time would be a distraction.

We emphasize the fact that coordinate is the number which is associated with a point on the line. Although we must be careful not to confuse coordinate of a point with the point itself, we will often say "the point 4" when we mean the point with coordinate 4. This is in keeping with the usual language of mathematics.

The distinction between number and name of a number comes up here for the first time.

Observe that the general statement concerning rational numbers is not a definition, because we do not at this point include the negative rational numbers. After we have introduced negative numbers in Chapter 6 we will have a definition of rational numbers. For the moment we want to have the idea that numbers like $\frac{2}{3}$, $\frac{5}{7}$, etc., are among the rationals. It is also possible to say that a number represented by a fraction indicating the quotient of a whole number by a counting number is a rational number.

A rational number may be represented by a fraction, but some rational numbers may also be represented by other numerals. The number line illustration gives the name "2" as well as the fractions "$\frac{4}{2}$", "$\frac{6}{3}$", "$\frac{8}{4}$" to name the number 2. The same diagram makes clear that not all rational numbers are whole numbers. The students may have seen some fractions that do not represent rational numbers; such as $\frac{\sqrt{2}}{2}$, $\frac{4\pi}{3}$, etc. So-called "decimal fractions" are not, by this definition, fractions.

It is necessary to keep the words "rational number" and "fraction" carefully distinguished. Later on in the course it will be seen that the meaning of the term "fraction" includes any expression, possibly involving variables, which is in the form of an indicated quotient.

The idea of "density" of the rational numbers is introduced. By "density" of numbers we mean that between any two numbers there is always another, and hence that between any two numbers there are infinitely many numbers. This suggests that, on the number line, between any two points there is always another point, and, in fact, infinitely many points.

Although it is assumed that the students are familiar with numbers like $\sqrt{2}$, π, etc., the discussion of irrational numbers is postponed until later in the course.

For the time being, until Chapter 6, we shall concentrate on the non-negative real numbers. This set of numbers, including 0 and all numbers which are coordinates of points to the right of 0, we call the set of numbers of arithmetic. After we establish the properties of operations on these numbers (in Chapter 4) we shall consider the set of all real numbers which includes the negative numbers (in Chapter 6). Then in Chapters 7, 8, 9 and 10, we spell out the properties of operations on all real numbers.

1-3. Addition and Multiplication on the Number Line

We use this graphical device to illustrate some of the properties of addition and multiplication of the real numbers. The question, "What number added to 5 is 8?" leads to the illustration of subtraction on the number line. We interpret "$8 - 5$" as meaning, "the number added to 5 is 8"; that is,

$$8 - 5 = n \text{ is the same as } 5 + n = 8.$$

The problem, $5 - 8 = n$, leads to the numbers to the left of 0. These problems are deferred until after the negative numbers are introduced.

The importance of the idea of closure under given operations follows naturally from the above problem, since the numbers of arithmetic are not closed under subtraction. We end this chapter with a few examples of sets that are or are not closed under given operations.

Chapter 2

NUMERALS AND VARIABLES

In this chapter we begin to make precise the language of algebra that we still use throughout this course. Again, much of this may be familiar to some of the students and they might well go through this chapter very quickly.

2-1. Numerals and Numerical Phrases

The aim of this section is to bring out the distinction between numbers themselves and the names for them and also to introduce the notion of a phrase. Along the way, a number of important conventions used in algebra are pointed out.

We do not want to make a precise definition of "common name". The term is a relative one and should be used quite informally. Note that some numbers, such as $\pi\sqrt{2}$, do not have what we would wish at this time to call a common name, while some may have several common names (e.g., $\frac{1}{2}$, 0.5 and $2\frac{1}{2}$, $\frac{5}{2}$, etc.).

The ideas of indicated sum and indicated product are very handy, particularly in discussing the distributive property, and will be used frequently.

The agreement about the preference for multiplication over addition is made to facilitate the work with expressions and not as an end in itself. In certain kinds of expressions the agreement might also apply to division as well as multiplication, for example, when division is written in the form $2/3$ or $2 \div 3$. We prefer to avoid these forms. We use the form $\frac{2}{3}$ whenever possible.

The use of parentheses might be compared to the use of punctuation marks in the writing of English. Emphasis should be placed on the use of parentheses to enable us to read expressions without ambiguity and not on the technique of manipulating parentheses.

The words "numeral" and "numerical phrase" denote almost the same thing. A phrase may be a more complicated expression which involves some operations; "numeral" includes all these and also the common names of numbers. The term "numerical phrase" is a useful term for a numeral which involves indicated operations.

The idea of writing $3 = 2$ as a sentence may seem odd. However, this is a perfectly good mathematical sentence which happens to be false. The concept of "true" or "false" sentences is an important one and becomes particularly fruitful when we discuss variables and open sentences in Chapter 3.

We have been doing two kinds of things with our sentences: We talk about sentences, and we use sentences. When we write

"$3 + 5 = 8$" is a true sentence,

we are talking about our language; when, in the course of a series of steps, we write

$$3 + 5 = 8,$$

we are using the language. When we talk about the language, we can perfectly well talk about a false sentence, if we find this useful. Thus, it is quite all right to say

"$3 + 5 = 10$" is a false sentence;

but it is far from all right to use the sentence

$$3 + 5 = 10$$

in the course of a proof. When we are actually using the language, false sentences have no place; when we are talking about our language, they are often useful.

The words "true" and "false" for sentences seem preferable to "right" and "wrong" or "correct" and "incorrect" because the latter often imply moral judgments to many people. There is nothing illegal, immoral, or "wrong" in the usual sense of the word about a false sentence.

2-2. Some Properties of Addition and Multiplication

The aim of this and the next section is to look at the fundamental properties of addition and multiplication in terms of specific numbers. It is important to emphasize the pattern idea here and we do this by writing, for example, the following for the associative property for addition:

(first number + second number) + third number =
first number + (second number + third number).

The use of the properties of addition and multiplication as an aid to computation in certain kinds of arithmetic problems is both interesting and important, but is not the main point of our discussion. The properties themselves play a fundamental role in this course.

After we extend the set of numbers to include the negative real numbers, we want to be sure we can define the operations of addition and multiplication for the set of all real numbers so that the properties of these operations are preserved. These operations should continue to be the familiar operations in the set of numbers of arithmetic.

Although we do not define binary operation explicitly--we do indicate that the operations of addition and multiplication are binary operations. That is, the sum of two numbers of arithmetic is a unique number of arithmetic and the product of two numbers of arithmetic is a unique number of arithmetic. From this we infer that closure of the set of whole numbers under these operations will also imply uniqueness of sum and product.

We shall return to the properties in Chapter 4 where general statements using variables will be given. They are discussed here not only as a part of the "spiral method" but because the distributive property is needed in introducing the concept of variable.

2-3. The Distributive Property

In this section we consider the distributive property of multiplication over addition. We do not continue to use the complete name but simply call this the distributive property. We verbalize this in terms of numbers of arithmetic and include true and false sentences. The aim of these, in the cases in which the sentences are true, is to have the student recognize their truth, not because both sides can be reduced to the same common name, but because the sentence is an example of a true pattern. The false sentences are intended to catch some mistakes students might make through misunderstanding of the distributive property, or misuse of the notation for mixed fractions.

2-4. Variables

The word "variable" has many connotations in mathematics and is, indeed, a difficult word to define. As it is used here, it represents an unspecified number from a given set of numbers. Thus, in the problem used to introduce the term "variable", the variable is used to represent a number from the set {1, ..., 100}. In other problems, the variable may represent an unspecified number of arithmetic or an unspecified whole number. The important idea here is that the variable always represents a number; that is, "x" or "n" or any letter used as a variable names a number from a given set.

The set of numbers from which the values of the variable are chosen is called the domain of the variable. In this part of the text, the domain is assumed to be the set of numbers of arithmetic, unless otherwise specified.

"Variable" will be discussed again in these Notes for Chapters 16 and 18.

Once variables have been introduced we distinguish between "numerical phrase" and "open phrase". This offers no trouble since an open phrase is defined to be a phrase containing one or more variables.

Note that we use the language, "if n is 7" or "let n have the value 7", rather than "n = 7". "n = 7" is an open sentence in our terminology.

Chapter 3

OPEN SENTENCES

3-1. Open Sentences

We call a sentence which involves a variable an "open sentence" since we cannot decide whether the sentence is true or false.

Once the domain of the variable is defined and the value is assigned to the variable, we can say either the sentence is true or the sentence is false. The values of the variable which make the sentence true are called "truth numbers". At this point we are interested in finding the truth numbers by "guessing". In later chapters we shall discuss techniques of finding truth numbers.

3-2. Truth Sets of Open Sentences

We call the set of truth numbers of an open sentence the truth set of the open sentence.

We also find truth sets of open sentences which are not equations. For example, the truth set of

$$3 + x \neq 6$$

is the set of all numbers of arithmetic except 3.

We graph the truth set of a given open sentence; i.e., we locate the points on the number line whose coordinates are the numbers in the truth set.

3-3. Sentences Involving Inequalities

A mathematical sentence may be of the form

$$3 < 4.$$

The meaning of "$<$" is derived from the position of the points on the number line. That is, "$3 < 4$" is true if and only if the point with coordinate 3 is to the left of the point with coordinate 4. The concept of "order relation" is deferred until Chapter 10 when we discuss this for the set of all real numbers.

In Items [illegible], the connection between subtraction and order is implied. This is in keeping with the spiral technique of this text. Subtraction is not discussed as an operation until Chapter 11.

The graph of the truth sets of inequalities presents no new problem.

3-4. Compound Sentences

The analogy between mathematical language and English is continued with the idea of compound sentences. The mathematical use of "or" should be particularly noted. This is the same as the legal term "and/or" rather than the exclusive "or" of ordinary English.

The discussion in the text may be summarized in the following table where p and q are mathematical sentences, and T and F mean true and false, respectively.

p	q	p or q	p and q
T	T	T	T
T	F	T	F
F	T	T	F
F	F	F	F

We also indicate the truth values for the negation of a compound sentence: If "p and q" is a false statement (that is, if "not p and q" is true), and if p is true, then q must be false.

These ideas are used in the proofs of theorems throughout the text both in the proofs of theorems and in the problems.

The truth set of a compound sentence with the connective "or" is the union of the truth sets of the simple sentences. The graph of the compound sentence is the set of points belonging to the graphs of each of the sentences.

The truth set of a compound sentence with the connective "and" is the intersection of the truth sets of the simple sentences. The graph of the compound sentence consists of those points which are common to the graphs of all the simple sentences.

Chapter 4

PROPERTIES OF OPERATIONS

This course is concerned with a systematic study of numbers and their properties. Arithmetic would seem at first to have had the same purpose, and in introducing this chapter we must pause and examine the difference. Arithmetic consists primarily of a rather mechanical application of a large number of rules for computing correctly, and very little attempt is usually made at any real understanding of these rules and of the numbers to which they apply. In algebra, on the other hand, we are interested in understanding rather thoroughly why numbers and the operations we do with them act as they do. We search here for general properties of the numbers and the arithmetic operations with which the student is already familiar. In short, we are interested in what mathematicians sometimes call the "structure" of the "system" of numbers.

It is inevitable that many of the general properties of numbers and of the operations we do on them are already quite familiar to the student from the study of arithmetic. The properties are not, however, familiar as explicit principles. Our aim in Chapter 1 was to have the student discover some of these properties through practice with examples. In this chapter the properties are formalized. The properties which we formalize in the mathematical language we have developed are the commutative and associative properties for both addition and multiplication, the distributive property, and what we call the addition property of 0 and the multiplication property of 1. We have also included the multiplication property of 0, although this could be deduced from the other properties. Later, in Chapters 5 and 7, we see that all these properties of the operations hold for all real numbers. Here we are considering only the non-negative reals; that is, the numbers of arithmetic.

A second goal of this chapter is to develop a good deal of technique in the simplification of algebraic expressions. We do this here because it is an essential feature of this course that the multitude of exercises required to gain manipulative facility be tied to the ideas from which the techniques derive their validity. We practice algebraic simplification at the point when the principles upon which the simplification rests are first developed, and many times thereafter. These principles, moreover, are precisely the properties of numbers which are stated in this chapter.

The order of Sections 4-1, 4-2, and 4-3 may seem strange, with the identity elements first, the closure properties second, and the associative and commutative properties third. It was chosen because the properties of 1 and 0 require only one variable in their statement, while the closure properties require two and the associative and distributive properties three. Thus, the properties which are easiest to state come first.

4-1. Identity Elements

The addition property of 0 states that 0 is the identity element for addition. Likewise, the multiplication property of 1 states that 1 is the identity element for multiplication. We prefer to state these as "two-sided" properties in order to save steps in proofs. We appeal to the commutative property for addition and for multiplication as a justification for the "two-sided" statement since these properties were "discovered" in Chapter 2. Later in this chapter, Section 4-3, we restate these properties in the language of algebra.

The multiplication property of 0 is not, like the others, a fundamental property of the number system, but it can be derived from the other properties. It is included here because we shall want to use it frequently.

The multiplication property of 1 is used to help simplify fractions. The student will remember from arithmetic that any rational number can be written in reduced form. This is equivalent to saying that $\frac{a}{b}$ and $\frac{k \cdot a}{k \cdot b}$ are names for the same number. In fact, this follows from the definition of multiplication of rational numbers; i.e., $\frac{k}{k} \cdot \frac{a}{b} = \frac{k \cdot a}{k \cdot b}$, and finally, the fact that $\frac{k}{k} = 1$ for any positive number of arithmetic k.

Later in the course we shall use these same ideas for all real numbers.

4-2. Closure

We have already stated that closure of a set under a given operation will imply uniqueness. That is, if a and b are numbers of arithmetic, $a + b$ is a unique number of arithmetic and $a \cdot b$ is a unique number of arithmetic.

4-3. Associative and Commutative Properties of Addition and Multiplication

What we have done here is translate the English statements of the associative and commutative properties obtained in Chapter 2 into the language of algebra. This is an example of a type of problem which is considered more systematically in Chapter 5. The comparison of the English statement with the algebraic statement shows the advantage of the latter in both clarity and simplicity.

We show other binary operations which are neither associative nor commutative.

A short section is given on preferred forms for numerals like

$$3abc \quad \text{and} \quad 3x^2y.$$

It is simply better form to write $3x^2y$, rather than $xy3x$.

4-4. The Distributive Property

This is the property that relates the two operations in which we have been interested. We state two forms of the distributive property; i.e.,

For all numbers of arithmetic a, b, and c:

$$a(b + c) = ab + ac$$
$$(b + c)a = ba + ca.$$

We then write the other two forms using the commutative property. We choose to write either of the above as the distributive property in order to save steps in proofs.

This property is used time and time again throughout this text and is of fundamental importance in much of the work that follows. The student will gain facility with use and should not be expected to attain full mastery at this time.

The terms "indicated product" and "indicated sum" have been introduced to help the student see the pattern of the distributive property as well as to see the difference between the forms

$$(x + y)(z + w)$$

and

$$xz + xw + yz + yw.$$

In this section the student gets a chance to use many of the properties to "simplify" an open phrase. For example, in simplifying

$$\begin{aligned} 2x + 3y + 4x + 6y &= (2x + 4x) + (3y + 6y) \\ &= (2 + 4)x + (3 + 6)y \\ &= 6x + 9y \end{aligned}$$

the student needs to use the commutative and associative properties of addition and the distributive property.

The pattern for writing

$$(2x + y)^2 = 4x^2 + 4xy + y^2$$

is given at this time; but the emphasis should be on the use of the distributive property rather than on the square of a binomial.

4-5. The Numbers of Arithmetic

Now that we have formalized the properties of addition and multiplication for the numbers of arithmetic, we stop to list these properties and to discuss the motivation of this text.

The system of the numbers of arithmetic would consist of the set of numbers, the operations on these numbers, and the properties which these operations enjoy. The structure of this system underlies the techniques of arithmetic with which the student is familiar.

We have studied only the properties of addition and multiplication although we have recognized that the operations of subtraction and division of the numbers of arithmetic are familiar to the student. Indeed, the positive numbers of arithmetic are also closed under division. In particular, if a is any non-zero number of arithmetic, there exists a number of arithmetic b such that $ab = 1$. This idea (of the multiplicative inverse) is discussed in Chapter [illegible] for all real numbers after we have extended our number set to include the negative real numbers. Once the multiplicative inverse has been defined, we are able to define division in terms of multiplication (Chapter 11) and we find that we do not need to consider any new properties for this operation.

Subtraction presents a different problem. The numbers of arithmetic are not closed under subtraction. Thus, if a and b are two distinct numbers of arithmetic, then one of the symbols $b - a$ or $a - b$ names a number of arithmetic. Another way to express this fact is that the solution set of the equation $a + x = 0$, for a, a non-zero number of arithmetic, is the null set. Once we extend our set to include the negative real numbers (Chapter 6) we can solve the equation $a + x = 0$, $a \neq 0$. We will call the numbers a and x additive inverses or opposites and we can then define subtraction in terms of addition (Chapter 11).

You will find a more complete discussion of the real number system in the notes for Chapter 10.

Chapter 5

OPEN SENTENCES AND ENGLISH SENTENCES

In this chapter we concentrate on the mathematical interpretation of verbal problems. This has long been recognized as one of the "stumbling blocks" in the mastery of algebra.

A "word problem" gives rise to the question: What number satisfies a certain set of conditions? In looking for the answer to this question, the procedure is as follows:

(1) Find an open sentence such that the number we are seeking is in the truth set of that sentence.

(2) Determine the truth set of the open sentence.

(3) Test the elements in the truth set in order to find the number which answers the question.

In this chapter we concentrate on the first step in this procedure.

There is one important point which must be made and is best illustrated by considering a particular problem.

Problem: The length of a rectangle is [illegible] feet more than the width. The area is [illegible] square feet. Find the width.

If w is the number of feet in the width of the rectangle; that is, if w is the number which satisfies the conditions of the problem, then $w(w + [illegible]) = [illegible]$ is, in fact, a true sentence.

If we are asked to find the number of feet in the width, then we are asked to find a number which is an element of the truth set of the open sentence

$$w(w + [illegible]) = [illegible].$$

In the open sentence "$w(w + [illegible]) = [illegible]$" the variable is w and the domain of w is the set of positive numbers of arithmetic. The number which we are seeking is an element of the truth set of the open sentence.

We are led, therefore, to the open sentence whose truth set contains the required number. In this chapter our interest is only to find the open sentence, not to find its truth set.

5-1. English and Mathematical Language

We introduce this section with a problem to illustrate the ideas of this chapter. [illegible] practice in finding interpretations for open phrases [illegible] from mathematics to English would lead to many different ideas in a classroom discussion and we try to build in some of this variety in the program.

We continue to emphasize the fact that a variable must represent a number.

5-2. English Phrases and Open Phrases

We reverse the procedure of the previous section and find the open phrases suggested by given word phrases. Thus, we are one step closer to the desired goal--open sentences suggested by word problems.

5-3. Interpreting Open Sentences

We continue the technique of translating from mathematics to English. Inequalities as well as equations are considered.

5-4. Writing Open Sentences

This section culminates the development. We first find the open phrases, and then express the relationships given by the problem to find an open sentence. Note that we do not try to give a series of rules for special types of problems such as mixture problems. We are interested in having the student find the relationships and express them, rather than in giving him strict type forms for problems.

The student should be conscious of the domain of the variable in the open sentences that arise from word problems. The domain is usually specified by the problem. For example, if a problem is about the number of students, the domain would be the set of whole numbers, whereas in the problem about the width of the rectangle, every non-zero number of arithmetic is in the domain of the variable.

5-5. Review

Note that this review is in programmed form.

INTRODUCTION TO PART 2

(Chapters 6 - 11)

Having "set the stage" by a rather careful study of the non-negative real numbers, drawing heavily on previous knowledge of arithmetic, we are now ready to examine the set of real numbers. Our procedure is to examine the negative half of the number line. We give names to the points to the left of zero. Our point of view is that we extend the already familiar set of numbers of arithmetic by attaching, or adjoining, the set of negative numbers. The resulting set is the set of <u>real numbers</u>. This approach has several advantages. We do not need to distinguish "signed" and "unsigned" numbers; for us the non-negative real numbers are the numbers of arithmetic. When the operations of addition and multiplication are defined, we need not prove that the familiar properties hold for the non-negative real numbers. In fact, we motivate our definitions by pointing out that these operations in the set of real numbers must yield results for the non-negative real numbers which are constant with our earlier experience.

After introducing the negative numbers we define addition, multiplication, and order in the next chapters. At the end of Chapter 10, we "take stock" of our development to that point. In Chapter 11 we once again "extend" our knowledge by defining subtraction and division in a manner analogous to their definition in arithmetic.

Chapter 6

THE REAL NUMBERS

6-1. The Real Number Line

We introduce the negative numbers in much the same way that we labeled the points on the right side of the number line, which correspond to the numbers of arithmetic. Our notation for negative four, for example, is $^-4$, and we definitely intend that the dash "$^-$" be written in a raised position. At this stage, we do not want the student to think that something has been done to the number 4 to get the number $^-4$, but rather that $^-4$ is a name of the number which is assigned to the point 4 units to the left of 0 on the number line. In other words, the raised dash is not the symbol of an operation, but only an identifying mark for numbers to the left of zero.

In Section 6-3, the student will be able to think of $^-4$ as the number obtained from 4 by an operation called "taking the opposite". The "opposite of 4" will be symbolized as -4, the dash being written in a centered position, and -4 will turn out to be a more convenient name for $^-4$.

Since each number of arithmetic has many names, so does each negative number. For example, the number $^-7$ has the names

$$^-\frac{14}{2},\quad ^-(7 \times 1),\quad ^-(\frac{3+2}{8} \times \frac{56}{5}),\quad \text{etc.}$$

Once the negative numbers have been introduced, we have the objects with which the student will be principally concerned throughout the next four years of his mathematics education. It is much too cumbersome to have to refer to "the numbers which correspond to all the points on the number line" or to "the numbers of arithmetic and their negatives", and so we use the customary name, the real numbers. No special significance should be attached to the word "real". It is simply the name of this set of numbers.

We do not wish to digress too far into irrational numbers. The proof that $\sqrt{2}$ is not rational will be given in Chapter 15. In the meantime we simply want the student to see an irrational number such as $\sqrt{2}$ or π and to be told that there are many more. We hope that he comes to realize that $\sqrt{2}$ is a number between 1 and 2, between 1.4 and 1.5, between 1.41 and 1.42, between 1.414 and 1.415, and so on.

6-2. Order on the Real Number Line

A more detailed treatment of "is less than" as an order relation is presented in Chapter 10. At this point we simply wish to rely on the intuitive geometric interpretation of "is less than" as "is to the left of on the number line". Thus, the concept of order is extended from our use of this concept in Chapter 3 for the numbers of arithmetic.

We introduce the comparison and transitive properties at this point. These properties could have been stated for the numbers of arithmetic at an earlier stage.

The comparison property here given is also called the trichotomy property of order. Notice that it is a property of $<$; that is, given any two different numbers, they can be ordered so that one is less than the other. When the property is stated we must include the third possibility that the numerals name the same number. Hence, the name "trichotomy".

Although "$a < b$" and "$b > a$" involve different orders, these sentences say exactly the same thing about the numbers a and b. Thus, we can state a trichotomy property of order involving "$>$" as:

For any number a and any number b,
exactly one of the following is true:

$$a > b, \quad a = b, \quad b > a.$$

If, instead of concentrating attention on the order relation, we concentrate on two different numbers, then either "$a < b$" or "$a > b$" is true, but not both.

Similarly, we state the transitive property for "$<$", reserving treatment of "$>$" until Chapter 10. If students inquire, they should be allowed to develop statements of the transitive property for other relations: "$=$", "$\leq$", "$\geq$". In later work the student will encounter still other relations with this property: "is a factor of" for positive integers, congruence in geometry, etc.

Item 61 asserts that if $a \leq b$ and $b \leq a$ are both true, we may conclude that $a = b$. This reasonable and innocent appearing conclusion turns out to be one of the most useful criteria for determining that two variables have the same value! In many instances in the calculus, for example, one is able to show by one argument that $a \leq b$ and by another that $b \leq a$. He is then able to conclude that $a = b$. Given two numerals, it is usually trivial to check whether or not they name the same number. In the case of two numbers, of course, we have complete information. It is only when our information about two "numerals" is incomplete that a statement like, "If $a \leq b$ and $b \leq a$, then $a = b$", can possibly be useful as a tool.

In Items 87-95 the student is asked to compare certain pairs of real numbers. The transitive property may prove useful, but the student should feel free to use all of his available methods of attack.

6-3. Opposites

We now point out that, except for 0, the real numbers occur as pairs, the two numbers of each pair being equivalent from 0 on the real number line. We call each number in such a pair the opposite of the other. 0 is defined to be its own opposite.

Having observed that each negative number is also the opposite of a positive number, it is apparent that we have no need for two symbolisms to denote the negative numbers. Since the lower dash "-" is applicable to all numerals for real numbers while the upper dash "$^-$" has significance only for points to the left of zero, we naturally retain the lower dash. There are other less important reasons for dropping the upper dash in favor of the lower: More care must be exercised in denoting negative fractions with the upper dash than with the lower; the lower dash is universally used, etc. Henceforth, then, negative numbers like $^-5$, $^-(\frac{3}{7})$, $^-\sqrt{2}$, and so on, will be written -5, $-\frac{3}{7}$, $-\sqrt{2}$, etc. Thus, we read "-5" as either "negative 5" or "opposite of 5". Notice that it is not meaningful to say "$\frac{1}{2}$ equals negative negative $\frac{1}{2}$"; rather say, "$\frac{1}{2}$ equals the opposite of negative $\frac{1}{2}$", or "$\frac{1}{2}$ equals the opposite of the opposite of $\frac{1}{2}$".

The student must learn to designate the opposite of a given number by means of the definition. We do not say and do not let the student say, "To find the opposite of a number, change its sign". This is very imprecise (in fact, we have never attached a "sign" to the positive numbers).

The student is well aware that the lower dash "-" is read "minus" in the case of subtraction. We prefer to retain the word "minus" for the operation of subtraction and not use it as an alternative word for "opposite of". Thus, the dash attached to a variable, such as "$-x$", will be read "opposite of".

If x is a positive number, then $-x$ is a negative number. The opposite of any negative number x is the positive number $-x$, and $-0 = 0$. Thus, the student should not jump to the conclusion that when n is a real number, then $-n$ is negative; this is true only when n is positive.

We do not like to read "$-x$" as "negative x" since $-x$ may, in fact, be positive. Some teachers read "$-x$" as the negative of x. In this usage the "negative of x" is synonymous with "the opposite of x". We prefer the latter. In any event, we avoid reading "$-x$" as "minus x".

6-4. Absolute Value

The concept of the absolute value of a number is one of the most useful ideas in mathematics. We will find an immediate application of absolute value when we define addition and multiplication of real numbers in Chapters 7 and 8. In Chapter 10 it is used to define distance between points; in Chapter 15 we define $\sqrt{x^2}$ as $|x|$; in Chapter 19 it will provide good examples of equations solved by squaring both sides. Through Chapters 20 to 23 absolute values are involved in open sentences in two variables and in Chapter 24 it gives us interesting examples of functions. In later mathematics courses, in particular in the calculus and in approximation theory, the idea of absolute value is indispensable.

We do not expect complete mastery of absolute value at this point. By the time the course is completed, the student should be thoroughly familiar with the concept.

The usual definition of the absolute value of the real number n is that it is the number $|n|$ for which

$$|n| = \begin{cases} n, & \text{if } n \geq 0 \\ -n, & \text{if } n < 0. \end{cases}$$

This is also the form in which the absolute value is most commonly used. On the other hand, since students seem to have difficulty with definitions of this kind, we prefer to define the absolute value of a number in such a way that it can be clearly pictured on the number line. We avoid allowing the student to think of absolute value as the number obtained by "dropping the sign". This way of thinking about absolute value, although it appears to give the correct "answer" when applied to specific numbers such as -3 or 3, leads to no end of trouble when variables are involved. Other less common names for absolute value are <u>numerical value</u>, <u>magnitude</u>, and <u>modulus</u>.

By observing that this "greater" of a number and its opposite is just the distance between the number and 0 on the real number line, we are able to interpret the absolute value "geometrically".

6-5. Summary and Review

Chapter 7

PROPERTIES OF ADDITION

7-1. Addition of Real Numbers

Having extended our set of numbers to include the negative numbers, we are now ready to define addition and multiplication in the set of real numbers.

From one point of view, it would be simpler to introduce multiplication of real numbers before addition. The definition of multiplication is simpler in terms of familiar operations. On the other hand, we have chosen to discuss addition first because we need the addition property of opposites in motivating the product of two negative numbers.

We first present an intuitively simple example (the profits and losses of an ice-cream vendor) which suggests how addition should be defined. We use the number line to illustrate how to obtain sums of real numbers, hinting at the formal definitions. Other examples hint at the properties of addition which are soon to be developed.

7-2. Definition of Addition

We review case-by-case how we add two real numbers on the number line and we lead the student to develop the desired definition of addition in the set of real numbers.

Several points should be noted. We do not wish students to memorize the definition. The formal statement is somewhat awkward but it is developed here to emphasize that in order to add two real numbers, what is needed is knowledge of:

1. How to add in arithmetic.
2. How to subtract a from b in arithmetic if $a \leq b$.
3. How to find the opposite [illegible] number.
4. How to find the absolute [illegible] a real number.

7-3. Properties of Addition

The student should be aware that our definition of addition stems from our intuitive feeling for the operation as shown in working with the number line. He should also see that our definition includes the familiar addition of numbers of arithmetic as a special case.

Since we have pointed out that the operation of addition in arithmetic has certain properties, it is natural to ask whether addition in the set of real numbers displays these same properties.

We use examples to make the statements of the commutative and associative properties reasonable. It is then asserted in the text that these properties could be proved from our definition of addition of real numbers and familiar properties of the numbers of arithmetic. The proof of the associative property requires the consideration of many cases and the proofs of certain of the cases involve ideas about subtraction which we have chosen not to develop at this point. The proof of the commutative property of addition is given here.

To show that $a + b = b + a$ for all real numbers a and b:

If $a \geq 0$ and $b \geq 0$, $a + b = b + a$ because they are numbers of arithmetic.

If $a < 0$ and $b < 0$,
$$a + b = -(|a| + |b|)$$
$$b + a = -(|b| + |a|)$$

But $|a|$ and $|b|$ are numbers of arithmetic, so $|a| + |b| = |b| + |a|$.

Therefore, $a + b = b + a$.

If $a \geq 0$ and $b < 0$,
$$\left.\begin{array}{l} a + b = (|a| - |b|) \\ b + a = (|a| - |b|) \end{array}\right\} \text{ if } |a| \geq |b|$$
$$\left.\begin{array}{l} a + b = -(|b| - |a|) \\ b + a = -(|b| - |a|) \end{array}\right\} \text{ if } |b| \geq |a|.$$

In either case $a + b = b + a$, since opposites of equals are equal.

If $a < 0$ and $b \geq 0$,
$$\left.\begin{array}{l} a + b = (|a| - |b|) \\ b + a = (|a| - |b|) \end{array}\right\} \text{ if } |a| \geq |b|$$
$$\left.\begin{array}{l} a + b = (|b| - |a|) \\ b + a = (|b| - |a|) \end{array}\right\} \text{ if } |b| \geq |a|.$$

In either case $a + b$ and $b + a$ name the same number.

A careful examination will show that we have considered every possible case, and every time we found

$$a + b = b + a.$$

Therefore, this is true for all real numbers a and b.

We next consider a property of addition of real numbers which is not analogous to properties of addition of numbers of arithmetic. This is the addition property of opposites.

To show that if a is a real number then $a + (-a) = 0$, we consider three cases:

If $a = 0$, then $-a = 0$ and the result is obvious.

If $a > 0$, then $-a < 0$ and $|a| = |-a|$. Hence, we apply part 3(a) of our definition of addition: $a + (-a) = |a| - |-a| = 0$.

If $a < 0$, then $-a > 0$ and $|a| = |-a|$. Hence, we apply part 4(a) of our definition: $a + (-a) = |-a| - |a| = 0$.

The last idea developed in this section is the addition property of 0:

For every real number a, $a + 0 = a$ and $0 + a = a$.

Proof: If $a \geq 0$, the result follows from the similar property of numbers of arithmetic:

If $a < 0$, then $|a| > |0|$ and part 4(b) of the definition of addition yields $a + 0 = -(|a| - |0|) = -|a| = a$.

$0 + a = a$ follows from the commutative property of addition.

7-4. Addition and Equality

If $a = b$ is true, we simply mean that a and b are names for the same real number. $a + c$ names a unique real number, hence, $b + c$ is simply another name for $a + c$. This is a formulation of the traditional "if equals are added to equals, the sums are equal". Since we have frequent occasion to use this generalization, we give it a name--the addition property of equality. The student should not conclude that this is a fundamental property of addition such as the commutative property. It is, as has been pointed out, a consequence of the uniqueness of the sum of two real numbers and the meaning of "=".

The major portion of the section is devoted to finding truth sets of simple equations. We introduce a slightly different language: "Determine the truth set", "determine the solution set", and "solve" all mean the same thing. By a "solution" of an open sentence we mean a member of its truth set.

We show the use of the various properties in solving equations, replacing one open sentence with another until we obtain a sentence whose truth set is obvious. Until the idea of the equivalence is well understood, we wish the student to be careful to stress this by pointing out that

"If ... is true for some x,
then ... is true for the same x", etc.

Careful, well organized work is emphasized, and the importance of checking is made clear. In Chapter 9 and again in Chapter 19 we present a thorough treatment of "equivalent sentences".

7-5. The Additive Inverse

In this section we present our first formal proof. The theorem that we have chosen to prove is a simple one--that the additive inverse of a real number is unique. We try to explain, in the program, our attitude towards "proof" at this stage of the student's mathematical training. The viewpoint about proofs in this course is not that we are trying to prove rigorously everything we say--we cannot at this stage--but that we are trying to give the students a little experience, within their ability, with the kind of thinking we call "proof". We don't wish to make a big issue of the idea of proof at this point and we don't expect all students to understand the concept the first time it is discussed. We hope that by the end of the year the student will have some feeling for deductive reasoning, a better idea of the nature of mathematics, and perhaps a greater interest in algebra.

We also lead the student to prove two other results:

(1) $-(a + b) = (-a) + (-b)$

(2) If $a + c = b + c$, then $a = b$.

The second of these is often called the cancellation property of addition. We do not use this terminology. (See also, these Notes, Section 9-2.)

In proving these theorems, the student gains more experience in developing proof and at the same time is shown an immediate use of the theorem previously proved.

7-6. Summary and Review Problems

Chapter 8

PROPERTIES OF MULTIPLICATION

8-1. Multiplication of Real Numbers

As with addition of real numbers, we wish to lead the student to the definition of multiplication of real numbers. We start with an example, again involving profit and loss, which perhaps provides some clue. We then build towards the formal definition, using the student's knowledge of multiplication in arithmetic. It is pointed out that since the set of numbers of arithmetic is a subset of the real numbers, our definition of multiplication of real numbers must give the familiar results when applied to the numbers of arithmetic. We stress that the definition should preserve, for the real numbers, those properties of multiplication which hold for the numbers of arithmetic.

How we use these ideas to motivate and to develop the definition of multiplication may be seen in the program. In order to apply the definition of the product of two real numbers a and b, the student needs to know (1) how to multiply two numbers of arithmetic ($|a|$ and $|b|$), and (2) how to decide whether the product is negative or non-negative.

Section 8-1 concludes with the theorem that $|ab| = |a|\cdot|b|$. This result is essential for the proofs of Sections 8-2 and 8-3.

8-2. The Commutative Property and the Multiplication Properties of 1 and 0

Having developed the definition of multiplication of real numbers from the point of view that we wish certain properties to be true, we now verify that, in fact, they are true.

We believe that most students will be able to understand the proofs of this section. Page 280 contains a statement of our attitude toward "proof" at this stage.

8-3. The Associative and Distributive Properties

The partial proof of the associative property presented here is somewhat difficult and we have indicated that the student may omit it. The proof is based on the fact that $|(ab)c| = |a||b||c| = |a(bc)|$. This result is proved in Items 7-14.

We present two proofs of $(-1)a = -a$. The proof on page 293 is optional. After we discuss the distributive property, an alternate proof is given.

Notice that we do not give a proof of the distributive property. In order to construct a proof we would have to consider many cases. In addition, we would have to use a fact which we have not discussed--in the set of numbers of arithmetic, multiplication is distributive over subtraction.

8-4. Use of the Multiplication Properties

This section provides practice in some of the necessary techniques of algebra. We wish to give sufficient practice, but we wish also to keep the techniques closely associated with the ideas on which they depend. We have to walk a narrow path between, on the one hand, becoming entirely mechanical and losing sight of the ideas and, on the other hand, dwelling on the ideas to the extent that the student becomes slow and clumsy in the algebraic manipulation. We hope the student understands that manipulation must be based on understanding. The right to skip steps and to compute without giving reasons must be earned by first mastering the ideas which lie behind and give meaning to the manipulation.

8-5. Summary and Review

Chapter 9

MULTIPLICATIVE INVERSE

9-1. Multiplicative Inverse

We first prove that every non-zero real number has a multiplicative inverse. The student already knows that every positive real number (a non-zero number of arithmetic) has a multiplicative inverse. Hence, it is only necessary to prove that every negative number has a multiplicative inverse. The proof depends on

(a) the definition of multiplicative inverse.

(b) the definition of opposite.

(c) Theorem 9-1: If a and b are positive real numbers and if $ab = 1$, then $(-a)(-b) = 1$.

Theorem 9-1 is, of course, a consequence of the definition of multiplication of real numbers.

We next consider the real number 0 and prove that 0 does not have a multiplicative inverse. This follows from the fact that $0 \cdot b = 0$ for all real numbers b, so that there cannot be a real number b such that $0 \cdot b = 1$.

The multiplication property of equality, which is useful in finding truth sets, is stated. This property, that if $a = b$, then $ac = bc$, is an immediate consequence of the meaning of "$=$" and the uniqueness of the product of two real numbers. Refer to the discussion of Section 7-2 on the addition property of equality.

Section 9-1 concludes with the proof that the multiplicative inverse of a non-zero real number is unique. The proof is analogous to the proof (in Section 6-5) that the additive inverse is unique.

9-2. Reciprocals

The name "reciprocal" is used as a synonym for "multiplicative inverse" just as "opposite" is another name for "additive inverse". The notation "$\frac{1}{a}$" is used for the reciprocal of a $(a \neq 0)$. Since the reciprocal is the multiplicative inverse, the results of Section 9-1 may be restated as:

(1) 0 has no reciprocal.

(2) The reciprocal of a non-zero real number is unique.

In the present section we prove several useful results:

(3) $\frac{1}{-a} = -\frac{1}{a}$ $(a \neq 0)$

(4) $\frac{1}{\frac{1}{a}} = a$ $(a \neq 0)$

(5) $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$ $(a \neq 0$ and $b \neq 0)$

Proofs of these are quite simple. They depend on the uniqueness of the reciprocal. Perhaps the student will verbalize these results: "The reciprocal of the opposite of a number is the opposite of its reciprocal", "The reciprocal of the reciprocal of a number is the number itself", "The product of the reciprocals of two numbers is the reciprocal of the product of the two numbers".

The most important theorem of this section is on page 325. The proof is in Items 79-84.

For real numbers a and b,
if $ab = 0$, then $a = 0$ or $b = 0$.

From the multiplication property of 0 it follows immediately that:

For real numbers a and b,
if $a = 0$ or $b = 0$, then $ab = 0$.

Combining these results, we state:

For real numbers a and b,
$ab = 0$ if and only if $a = 0$ or $b = 0$.

A brief discussion of the use of the terminology "if and only if" is provided. The important idea here is that, in proving an "if and only if" theorem, there are two things to be proved.

Item 109 challenges the student to try to prove the theorem:

If a, b, c, are real numbers and
if $ac = bc$, and $c \neq 0$, then $a = b$.

This theorem is often called the cancellation property of multiplication. We do not introduce this name because we wish to avoid the well-known misuses of "cancellation".

It is of interest to note that the cancellation property of multiplication is equivalent to the theorem "if $ab = 0$, then $a = 0$ or $b = 0$". That is, each may be proved using the other.

Assuming the cancellation property, we would argue as follows in order to prove: if $ab = 0$, then $a = 0$ or $b = 0$.

> Suppose $ab = 0$. Either $b = 0$ or $b \neq 0$. If $b = 0$, there is nothing more to prove, so assume $b \neq 0$.
> Then, $ab = 0 \cdot b$ since $0 \cdot b = 0$.
> We have $a = 0$ from the cancellation property, since $b \neq 0$.

The proof of the cancellation property given in the Answer Key uses multiplication by the reciprocal of a non-zero real number since we are dealing with reciprocal at this point. Here is an alternate proof:

Suppose	$ac = bc$ and $c \neq 0$.	
Then	$ac + (-bc) = 0$	addition property of equality
	$\big(a + (-b)\big)c = 0$	distributive property
Therefore,	$a + (-b) = 0$, since $c \neq 0$	
	$a = -(-b)$	uniqueness of additive inverse
	$a = b$.	

Notice that the cancellation property may be stated as follows: If $ac = bc$, then $a = b$ or $c = 0$. It is in this form that the result is restated in Chapter 19 and used in finding solutions of equations.

9-3. Solution of Equations

To this point we have only been able to solve equations of the form $x + b = c$ using the addition property of equality. In this section, using the multiplication property of equality, we are able to solve equations of the form $ax = c$ $(a \neq 0)$ and $ax + b = c$ $(a \neq 0)$.

It should be pointed out that we encourage the student to consider alternate procedures in finding solution sets. For example, to solve

$$56 = 2y + (-18)$$

it is perfectly proper to begin by adding (-56) to both sides, by adding 18 to both sides, or by multiplying both sides by $\frac{1}{2}$. In some cases certain procedures lead to more awkward arithmetic than others. It is only through practice that the student learns that the "best" first step in solving

$$\frac{x}{3} + \frac{2}{9} = \frac{3}{2} \text{ is to multiply both sides by } 18.$$

On the other hand, to solve

$$\frac{x}{3} + [illegible] = [illegible]$$

one probably should not start by multiplying both sides by 3.

The major portion of the section deals with the idea of <u>reversible steps</u> in solving an equation. We define two sentences to be <u>equivalent</u> if they have the same truth set. The discussion, in the program from page 332 to 338, is quite important. Remember, however, that this matter of equivalent sentences will be encountered throughout the remainder of the program. In particular, Chapter 1[illegible] is devoted to this topic. Here, as elsewhere in the course, we do not expect complete mastery of an idea the first time it is introduced.

One idea may bother the students. What is the truth set of $0 = 0$? This equation is equivalent

$$\text{to} \quad x + 0 = x + 0$$

$$\text{and, hence, to} \quad x = x.$$

Now $x = x$ has as its truth set the set of all real numbers. We may conclude that $0 = 0$ is also true for all real numbers.

In a similar fashion, we agree that the truth set of, say, $1 = 0$ is $\emptyset$. ($1 = 0$ is equivalent to $x + 1 = x$.)

9-4. Summary and Review

Chapter 10

PROPERTIES OF ORDER

10-1. The Fundamental Properties

Beginning in Chapter 1 we have assumed that the student has an intuitive feeling about the order of two real numbers in terms of the number line. In this section, we shift our point of view slightly. We look on "$<$" as a mathematical subject, an order relation. The relation is essentially defined by the properties we assign to it: the comparison, transitive, and addition properties of "$<$".

We then provide the student with numerous exercises which give practice in applying these properties in various situations.

Notice that while it is true that "$a < b$" and "$b > a$" say the same thing about the two numbers a and b, we concentrate, in this section, on the relation "$<$". "Greater than" is also an order relation but we prefer to treat it after we have developed further familiarity with "$<$".

10-2. Further Properties of Order

In this section we state and prove several consequences of the fundamental properties of $<$. These are

Theorem 10-2a. If $x + y = z$ and y is positive, then $x < z$.

Theorem 10-2b. If $x < z$, then there is a positive number y such that $x + y = z$.

Theorem 10-2c. If $a < b$, then

$$ac < bc \text{ if } c \text{ is positive,}$$
$$bc < ac \text{ if } c \text{ is negative.}$$

This last theorem is called the multiplication property of order and is used extensively in Section 10-3 and later in the course.

10-3. Solving Inequalities

We now remind the student that we could restate the results of Sections 10-1 and 10-2 in terms of "$>$". In particular, we may change from one order relation to the other.

To solve $-2x < 6$, for example, we may obtain the equivalent inequality $-3 < x$ from the multiplication property of "$<$". It is usually easier to interpret this conclusion in the form $x > -3$.

We provide a good deal of practice with solving inequalities of the general form

$$ax + b < c \quad (a \neq 0).$$

Once again, it should be remembered that the topic of solving inequalities will be considered in much more detail in Chapter 19.

The student is reminded that there is more than one way to start solving an inequality.

10-4. Review

10-5. The Real Numbers - Summary

We pause in our work to summarize for the student our development up to this point. In this section we are trying to shift the student's thinking about real numbers from a primarily inductive to a deductive point of view. The deductive point of view has, of course, been uppermost in our own minds from the beginning and has had a great deal of influence in our study of the real numbers. Thus, we have consistently worked toward the basic properties listed here. The idea to which the student should be exposed is that we could have started with these properties in the first place by considering them, not as a description of, but rather as a definition of, the real number system. Thus, "real number", "addition", "multiplication", and "order relation" become undefined terms, the fundamental properties become axioms (for an ordered field), and all other properties, as consequences of the axioms, become theorems. This is the approach which becomes commonplace in more advanced mathematics. Traditionally, the student first encountered a deductive approach in high school geometry.

Although we have given considerable preparation for the shift in point of view indicated here, most students obviously will not be able to appreciate its significance fully at this time. We have, moreover, no intention of proving everything from here on, but will continue as we have in the past, "discovering" properties and giving an occasional simple proof. The principal change is in our attitude toward the properties discovered: namely, that they could be proved if we had the time and experience to do so.

The fourteen fundamental properties listed also hold in the set of rational numbers. That is, the rational numbers, with the usual definitions of addition, multiplication, and order, also form an ordered field. The missing fundamental property, which distinguishes the set of real numbers from the set of rational numbers is called the completeness axiom. It can be stated in several ways, one being in terms of "least upper bounds". Before stating it, we first define an "upper bound" of a set as follows:

> Let S be a set of real numbers and b a real number such that $s \leq b$ for every s in S. Then b is called an <u>upper bound</u> for S. If there does not exist an upper bound for S which is <u>less</u> than b, then b is called a <u>least upper bound</u> for S.

We can now state the

> <u>Completeness Axiom</u>. If S is any set of real numbers for which there is an upper bound, then there exists a least upper bound for S.

The completeness axiom is needed, for example, to prove the existence of $\sqrt{2}$. In other words, one cannot prove, using <u>only</u> the fourteen properties, that there is a real number a for which $a^2 = 2$. As a matter of fact, the completeness axiom insures the existence, not only of real numbers like $\sqrt{2}$, but also of real numbers such as π, $\log 2$, $\sin \frac{\pi}{9}$, $e^{1.2}$, which the student will encounter in later courses. The completeness axiom guarantees that there are "enough" real numbers to fill in the number line "completely". The only irrational real numbers that play an important role in this course are square roots. We will return to the discussion of completeness in the notes for Section 15-2.

Chapter 11

SUBTRACTION AND DIVISION

11-1. Definition of Subtraction

At first glance our definition of subtraction does not appear to be a definition--at least, it is not in a form with which the student is familiar. It should be viewed in connection with the material that follows the definition.

If we start with two real numbers a and b, then the equation $b + n = a$ has a unique solution; namely, $a + (-b)$. Our definition asserts that $b + n = a$ and $a - b = n$ are equivalent. That is, $a - b$ is defined to name a unique real number.

We do not expect that the student will confuse the different uses of the "-" symbol. In fact, he will find it easier to write $4 - 3$ than $4 + (-3)$.

It will be noticed that we have avoided using the word "sign" for the symbols "+" and "-". We do not need the word, and since its misuse in the past has caused considerable lack of understanding (in such things as "getting the absolute value of a number by taking off its sign") we prefer not to use it.

A related point that we should mention is that we do not write $+5$ for the number five. The positive numbers are the numbers of arithmetic. We therefore do not need a new symbol for them. Thus, we write 5, not $+5$, and the symbol "+" is used only to indicate addition.

11-2. Properties of Subtraction

We used the fact that students were familiar with subtraction in arithmetic in formulating our definition of addition for the real numbers. We have now defined subtraction of real numbers in terms of addition of real numbers. The reason for this seeming inconsistency is that, at the beginning of the course, we wished to build on the student's arithmetical background. Our inductive development of the axioms of Section 10-5 was based on that background. Now we are assuming the axioms of Section 10-5 and shall develop our algebra on the basis of those properties. From the present point of view, subtraction is not an essentially distinct operation, it is simply a convenient and a conventional method of dealing with $a + (-b)$. Therefore, the "properties" of subtraction developed in this section are not new basic properties of the real number system but are, rather, consequences of the definition of subtraction.

11-3. Subtraction and Distance

The relation between the difference of two numbers and the distance between their points on the number line is introduced here to make good use again of the number line to help illustrate our ideas.

Students may have difficulty in solving inequalities such as "$|x - 3| < 5$". In this section we wish the student to reason as follows: "If a is a solution of $|x - 3| < 5$, then a must be less than 5 units from 3 on the number line. Therefore, the solution set is the set of numbers between -2 and 8".

Another source of difficulty for the student is learning to interpret $|x + 4|$ as the distance between x and -4.

We treat inequalities involving absolute value again in Chapter 29.

11-4. Division

Our treatment of division is parallel to our treatment of subtraction. The definition of division leads to the statement that, if $b \neq 0$, then "$\frac{a}{b}$" and "$a \cdot \frac{1}{b}$" name the same number. Division is related to multiplication in much the same way as subtraction is related to addition. The usual material on fractions is found in Chapter 13. The division algorithm for integers is discussed in Chapter 18.

11-5. Summary and Review

INTRODUCTION TO PART 3

(Chapters 12-17)

We now have completed our development of the real number system. In Part 3 we turn to consideration of specific types of algebraic expressions. Before doing so, we discuss, in Chapter 12, the factorization of positive integers.

It is not our intention to delve deeply into the study of the positive integers. We merely consider some of the properties that are useful in the further development of algebra. Most students have a good deal of intuitive understanding of the positive integers and will, most likely, find the discussion in Chapter 12 interesting and profitable.

The concepts and results of Chapter 12 are applied throughout the rest of Part 3. The usual manipulative skills in working with fractions, exponents, and radicals are developed in Chapters 13-15. At every opportunity we stress both the need for developing skill and the necessity for understanding the reasons which justify the manipulations. Particularly interesting is the material in Chapter 15 on the irrationality of $\sqrt{2}$ and on the iteration method of approximating square roots.

Chapter 16 and 17 deal with polynomials, with emphasis on quadratic polynomials. We stress the use of the distributive property in developing the usual factoring techniques. "Completing the square" of a quadratic polynomial is discussed at some length since this provides a general method of factoring and is to be used in later work with graphing parabolas. Chapter 17 concludes with a treatment of quadratic equations. The quadratic formula is developed in Section 17-4.

Chapter 12

FACTORS AND DIVISIBILITY

Basic to the material of this chapter are two theorems which are stated without proof.

The first of these is the Fundamental Theorem of Arithmetic, which states that a positive integer greater than 1 is either a prime or can be expressed as a product of primes in essentially only one way.

The second major theorem which we will need is the division algorithm: If n and d are positive integers, then there exist integers q (the quotient) and r (the remainder) such that

$$n = qd + r, \quad 0 \leq r < d.$$

In order to prove these theorems we would need to develop more completely the properties of the positive integers. In particular, we would need to discuss the so-called Well-Ordering Principle: Every non-empty set of positive integers contains a smallest element. Closely related is the principle of mathematical induction. Treatment of these ideas would take us too far afield.

The Fundamental Theorem of Arithmetic is used extensively throughout the remainder of the course. The division algorithm is treated more fully in these Notes for Chapter 18.

Let us emphasize that although the material of Chapter 12 is interesting and valuable for its own sake, our primary concerns are:

(1) to develop certain skills which are useful later.

(2) to point out certain ideas about factoring positive integers which turn out to be analogous to ideas about factoring polynomials.

Students who have studied the SMSG 7th grade course will have had a good start with many of the ideas of Chapter 12. They will probably need less time on this material than other students.

12-1. Factors

We say: The positive integer m is a factor of the positive integer n if there is a positive integer q such that mq = n.

If m is 1, then q is n, while if m is n, then q is 1. If m is not 1 or n, then we say that m is a proper factor of n.

12-2. Tests for Divisibility.

Tests for divisibility by 2 and 5 are reviewed and tests for 3, 4, 6, and 9 are developed in this section to help students in recognizing factors of integers. Notice that these tests are based on our decimal system of notation. Some students who have studied numeration using bases other than 10 may enjoy considering the possibilities of developing divisibility tests in other numeration systems.

In this section the student sees for the first time an indirect argument in the proof that if the square of an integer is even, then the integer is even. A second example of indirect proof occurs in Section 12-4, where we show that if a is a factor of b and a is not a factor of $b + c$, then a is not a factor of c. (a, b, c are positive integers.) Although the idea of indirect proof will occur again in later chapters, it is important that enough attention be paid to it here to enable the student to gain some understanding of it.

12-3. Prime Numbers and Prime Factorization

To insure understanding here and later, it should be stressed that when we are interested in factorization, we must explain what kinds of numbers we permit as factors. If we are considering only integers, then we say that 6 is a factor of 18, because $6 \times 3 = 18$ and 3 is an integer. We do not regard 7 as a factor of 18, even though $7 \times \frac{18}{7} = 18$, because $\frac{18}{7}$ is not an integer.

It is not very interesting to talk about factoring in the set of rational numbers, since if a and b are rational, and $b \neq 0$, we can always find a rational number c for which $a = bc$.

When we come to factoring polynomials we will use again the idea that what we can say about factoring an expression depends on the kinds of factors we permit.

In discussions of factoring we also need to have in mind some idea of when the way in which two factorizations differ is an interesting way. Consider again factorizations in the set of integers. Some of the ways of writing 6 as a product of positive integers are:

$$2 \times 3, \quad 3 \times 2, \quad 1 \times 2 \times 3, \quad 1 \times 6, \quad 6 \times 1.$$

Because of the commutative property (in products of several factors, the commutative and associative properties) we can always change the order of the factors in a product. We would consequently regard 2×3 and 3×2 as factorizations that are not different in an interesting way. Finally,

$1 \cdot a = a$ for all real numbers; hence, there is nothing especially interesting about $1 \times 2 \times 3$ as a different name for 2×3.

If we are asked, then, for the ways in which 6 can be expressed as a product of <u>two</u> factors, we regard 2×3 and 6×1 as the only essentially different ways.

Notice that according to our definition of prime, number 1 is <u>not</u> a prime number. Students sometimes question this. They should be reminded that in defining a new term we choose the definition that is most useful. If we were to call 1 a prime number, then 2×3, $1 \times 2 \times 3$, $1 \times 1 \times 2 \times 3$, etc. would all be distinct prime factorizations of 6. The Fundamental Theorem of Arithmetic would need to be reworded in an awkward fashion.

You may have students who wish to use exponents in writing products, and there is no reason why they should not do so if they are able to manage successfully. Some students will have learned about positive integer exponents in the 7th or 8th grade. Others may recognize possibilities of using them simply on the basis of their experience in this course. If they do, they are using good mathematical insight, and they should be encouraged.

Students can be led to understand the Fundamental Theorem by noting that they always obtain the same prime factorization of an integer, regardless of the order of steps they use to find it. Thus, in factoring 120, we may begin with the smallest factor (2, in this case) and write $120 = 2 \times 60 = 2 \times 2 \times 30$, etc. We may also note at once that $120 = 10 \times 12$, and then factor the 10 and 12. The same prime factorization results in either case.

In factoring integers, some students may prefer at first to handle each problem systematically, beginning with the smallest prime factor. With experience they will tend to look for shortcuts (like using the fact that $120 = 10 \times 12$) and they should be encouraged to do so. The Fundamental Theorem, in other words, should be understood not only as an important theorem about integers but as a theorem which we use frequently.

You should not encourage the student to feel that the Fundamental Theorem of Arithmetic is obvious. There are many algebraic systems in which this theorem does not hold. Consider, for example, the set of even positive integers $\{2,4,6,8, \ldots\}$. If multiplication in this set is defined in the obvious way, it would be logical to accept 6 as "prime" since it is not the product of two even numbers. Thus, 6, 10, 14, ... would be "prime", but 8, 12, 16, ... would not be. In this arithmetic, 6×6 and 2×18 would be two different prime factorizations of 36.

A proof of the Fundamental Theorem of Arithmetic can be found in many books on elementary number theory. (See Teachers References: Number Theory, Unique Factorization.)

12-4. Some Facts About Factors

Note the indirect proof used in Theorem 12-4b. For all students, work with factorizations of integers will be a good basis for factoring polynomials. More capable students should be made aware that the results in this section will help them find interesting shortcuts when they are factoring polynomials.

12-5. Summary

Chapter 13

FRACTIONS

A fraction is a numeral which has the form of an indicated quotient. The term "fraction", then, refers to the form of a numeral. We call 3, $\frac{3}{1}$, $2 + 1$, names (numerals) for the same number. Of these numerals, $\frac{3}{1}$ is a fraction. Similarly, $\frac{1}{4}$ and $.25$ name the same number, and $\frac{1}{4}$ is a fraction name for this number.

We do not want to be over-pedantic about terminology, however, and where no confusion results we sometimes use "fraction" to refer to a number. It would be unwise to insist that students always use rigorously precise language in speaking about fractions. However, they need to understand the distinction between a number (which has many names) and a name that has a particular form.

For example, it is not strictly correct to speak of "adding fractions". We should say, instead, "adding numbers which are represented by fractions". We do not want to insist on strict adherence to this awkward, though accurate, wording. Similarly, we may refer to the numerator and denominator of a fraction as numbers, though strict precision would call for the phrases, "numeral which is the numerator", etc.

We mention the word "ratio" as part of the language in certain applications. You may wish to mention to the students that an equation such as $\frac{x}{1197} = \frac{z}{19}$, which equates two ratios, is often called a "proportion". It seems undesirable at present to digress into a lengthy treatment of ratio and proportion, since it would be just a matter of giving special names to familiar concepts.

13-1. Multiplication of Fractions

In this and following sections we are simplifying fractions; that is, finding simplest numerals. We use the following conventions.

(1) There should be no indicated division if it can be avoided.

(2) If the simplest numeral must contain an indicated division, then the resulting expression should be written in "lowest terms".

(3) We prefer writing $-\frac{a}{b}$ to either of the forms $\frac{-a}{b}$ or $\frac{a}{-b}$.

We show that $\frac{ak}{bk} = \frac{a}{b}$ if b and k are any non-zero real numbers. It is easy to do this, using the definition of $\frac{a}{b} \cdot \frac{k}{k}$ and the multiplication property of 1.

You should recall that in the students' first experiences with fractions, in the elementary grades, he saw concrete examples illustrating such sentences as $\frac{1 \cdot 2}{2 \cdot 2} = \frac{1}{2}$; $\frac{3}{4} = \frac{6}{8}$, etc. We have assumed that the student entered this course with some knowledge about the non-negative rational numbers. We would assume that he knows already that if a, b, and k are positive integers, then $\frac{a}{b} = \frac{ak}{bk}$ is true. He may be well aware that the statement is true for numbers of arithmetic (if $b \neq 0$ and $k \neq 0$). The importance of the result stated above is that we extend the known pattern, showing that it applied whenever a, b, k are any real numbers and $b \neq 0$ and $k \neq 0$.

In other words, we are showing once again that a pattern already known for certain numbers is applicable for all real numbers (bearing in mind the need to avoid denominators which are 0).

The same comments apply to the other theorems of this chapter.

In applying the theorem that $\frac{ak}{bk} = \frac{a}{b}$, it is natural to use the language of "common factor" developed in Chapter 12. Thus, in simplifying $\frac{5x}{6x}$, we may call x a "common factor". In Chapter 16 we will discuss in more detail the ideas related to factoring expressions involving a variable. However, we use the language of factoring here whenever it is helpful.

We have avoided the word "cancel" because it is often used without understanding. The student who has learned to use this word should not be told it is wrong, but he should be required to explain the process he is using in other words to be sure he understands it.

13-2. Division of Fractions.

Three methods of simplifying "complex fractions" are given. In order to simplify $\frac{\frac{4}{3}}{\frac{5}{8}}$ we may write:

(1) $\frac{\frac{4}{3}}{\frac{5}{8}} = \frac{4}{3} \div \frac{5}{8} = \frac{4}{3} \cdot \frac{8}{5} = \frac{32}{15}$; or

(2) $\frac{\frac{4}{3}}{\frac{5}{8}} = \frac{\frac{4}{3} \cdot 24}{\frac{5}{8} \cdot 24} = \frac{32}{15}$; or

(3) $\frac{\frac{4}{3}}{\frac{5}{8}} = \frac{\frac{4}{3} \cdot \frac{8}{5}}{\frac{5}{8} \cdot \frac{8}{5}} = \frac{\frac{32}{15}}{1} = \frac{32}{15}$

The student should understand all three methods. He may wish to use a single method in all cases, or he may prefer to select for each problem the method that seems simplest.

13-3. Addition and Subtraction of Fractions

We wish to apply the prime factorization of integers to the problem of finding the least common multiple of the denominators. We do not want blind adherence to the method developed, however, but we do want to give the student a systematic way of approaching the problem. For example, if the student were asked to add the fractions $\frac{1}{2} + \frac{1}{6}$, the least common denominator can be quickly determined by inspection, and the student should do it this way. If, however, he is asked to add $\frac{1}{57} + \frac{1}{95}$, it may not be easy to determine the least common denominator by inspection. But by prime factorization

$$57 = 3 \cdot 19,$$
$$95 = 5 \cdot 19,$$

and the least common denominator is $5 \cdot 3 \cdot 19$.

It is good technique, both here and in later work on factoring, to leave expressions in factored form as long as possible. If this is done, simplifying is made easier.

13-4. Summary and Review

Chapter 14

EXPONENTS

14-1. Introduction to Exponents

Be careful to define a^n where n is an integer greater than 1, as a product in which a is used as a factor n times. The statement, "a is multiplied by itself n times" is not precise and ought to be avoided. (Notice, for example, that $a^2 = a \cdot a$, whereas "a multiplied by itself twice would mean $(a \cdot a) \cdot a$.) Notice that we also define a^1 as a.

14-2. Positive Integers as Exponents

In this section we derive the rules for working with positive integer exponents. That is, we show that for such exponents $a^m \cdot a^n = a^{m+n}$. We also show that

$$\text{if } m > n, \text{ then } \frac{a^m}{a^n} = a^{m-n};$$

$$\text{if } m = n, \text{ then } \frac{a^m}{a^m} = 1;$$

$$\text{if } m < n, \text{ then } \frac{a^m}{a^n} = \frac{1}{a^{n-m}}.$$

Students should not use these rules mechanically. They may prefer, for example, to simplify $\frac{x^7}{x^3}$ by remembering the meaning of x^7 and x^3. On the other hand, it is important that the student be well aware of these rules in their general form, since they will be extended in the sections that follow.

14-3. Non-Positive Integers as Exponents

In this section we define a^{-n}, where n is a positive integer and $a \neq 0$ as $\frac{1}{a^n}$. Thus, $a^{-3} = \frac{1}{a^3}$. We also define a^0 as 1 if $a \neq 0$.

Negative exponents will be used in the next chapter in connection with finding square roots. Students are also likely to need them in connection with "scientific notation" for writing numbers.

Students should understand that if we have defined a^n only for values of n which are positive integers, we are still free to assign any meaning we like to the symbols a^0, a^{-1}, etc. We do not prove that a^0 must be 1, for example. We simply show that if we define a^0 as 1 (if $a \neq 0$), then we can say $\frac{a^m}{a^m} = a^{m-m}$. Our wish to have this statement true <u>motivates</u> the definition. The logic here is analogous to that of our discussion in Chapters 6 and 7, of the definition of $a + b$ and ab, where a and b are real numbers.

The student should understand that when we have defined a^{-n} for positive integers n, we have defined a^n for <u>all</u> integers in such a way that:

$$a^m \cdot a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$a^{-n} = \frac{1}{a^n}.$$

For example, $a^m \cdot a^n$ was proved (Section 14-2) for positive integer values of m and n. The student should recognize two things: (1) We cannot be <u>sure</u> that $a^{-3} \cdot a^{-2} = a^{(-3)+(-2)}$, for example, without further discussion; (2) We might <u>suspect</u> that it is true, because we have often found in the past that patterns that hold for some numbers can be extended to apply to others.

Thus, all students should understand why we reconsider the statements $a^n = \frac{1}{a^{-n}}$, $a^m \cdot a^n = a^{m+n}$, and $\frac{a^m}{a^n} = a^{m-n}$, <u>proving</u> that each is true for all integer values of m and n (excluding, when necessary, the case where a is 0). Though <u>all</u> students should understand <u>why</u> the proofs are included, you may wish to omit, with less capable students, the details of the proofs.

14-4. Using Exponents

We complete our list of rules about exponents with

$$(a^m)^n = a^{mn}.$$

We now have all the major results about exponents and we can apply them to all integer exponents.

Students should be reminded frequently that there is often more than one way to solve a problem. Less able students may find that they make fewer errors if they use a single pattern persistently, and they should be permitted to do so. However, they should not regard theirs as the <u>only</u> possible method. More capable students may enjoy looking for different ways to solve a problem. Consistent with this philosophy, the program in Section 14-4 presents alternative solutions to some examples.

14-5. Summary and Review

Chapter 15

RADICALS

Having prepared the way by factoring integers and studying exponents, we proceed to a study of radicals.

15-1. Roots

We define $\sqrt{b}$, where b is a non-negative real number, as the <u>non-negative</u> number whose square is b. Under this definition, $\sqrt{b}$ is a symbol which names exactly one real number if $b \geq 0$. Verbal usage is sometimes less precise. In later courses, for example, students may speak of "the two square roots of 2", or, after they have complex numbers at their disposal, "the three cube roots of 1", etc. We do not want to introduce awkward or over-restrictive verbalisms, but we want to stress the fact that "$\sqrt{b} = a$" means "$a^2 = b$ <u>and</u> $a \geq 0$". This usage is entirely consistent with later ones.

When there is a variable under the radical, we must be careful in two ways. First, since we are considering real numbers, $\sqrt{x}$ is <u>meaningless</u> if $x < 0$. Consequently, when a variable occurs under the radical we must restrict its domain so as to avoid negative values of the radicand.

Second, in simplifying an expression like $\sqrt{x^2}$ we must remember that our <u>result</u> must be non-negative for all admissible values of the variable. Thus, we cannot write $\sqrt{x^2}$ as simply x, since $\sqrt{x^2} = x$ is not true for negative values of x. $(\sqrt{(-3)^2} \neq -3)$. We may write $\sqrt{x^2} = |x|$. On the other hand, we may write $\sqrt{x^3} = x\sqrt{x}$ since $\sqrt{x^3}$ and $\sqrt{x}$ are defined only for non-negative values of x, and for these values the statement is true.

We may write $\sqrt{x^4} = x^2$ since x^2 is non-negative for all real values of x. The student should not be encouraged to memorize cases, but should analyze such examples on their merits, referring always to the definition.

15-2. Irrational Numbers

In this section we prove that there is no rational number whose square is 2. Our proof that $\sqrt{2}$ is irrational is not the only one we might have given. It has the merit of beginning, essentially, with the fact that the square of an odd number is odd and the square of an even number is even.

You will find further material in the existence of $\sqrt{2}$ and on the fact that it is irrational in <u>Numbers: Rational and Irrational</u>. (See references.)

Some students may raise questions about whether "there is such a number as $\sqrt{2}$". Though perhaps awkwardly stated, these questions are worth some discussion. The following ideas are relevant.

When we define $\sqrt{a}$, where $a \geq 0$, as the non-negative real number whose square is a, we tacitly make two assumptions:

1. There is a non-negative real number whose square is a.
2. There is not more than one such number.

The fact that there is not more than one non-negative real number whose square is a follows immediately from the fact that if b and c are non-negative, and $b < c$, then $b^2 < c^2$. (This is proved in Items 10 to 14, Section 15-2.)

We have assumed from the beginning of this course that $\sqrt{2}$ is a real number. In Chapter 6 we saw a geometric construction for locating on the number line a point corresponding to $\sqrt{2}$. We hope that the student has the intuitive feeling that since there is a point on the number line such that the square of its distance from the origin is 2, there is a real number, corresponding to this point, such that the square of this number is 2. In particular, students who have used SMSG materials in the elementary grades should have become familiar with the idea that if we can locate a point on the number line, then we can identify a real number with this point.

As we noted earlier, we cannot prove that there is a real number whose square is 2 by using only the basic properties listed in Section 10-5. The Completeness Axiom (see Teachers' Notes, Section 10-5) is needed to insure the existence of $\sqrt{2}$, $\sqrt{3}$, etc.

Students should not form the mistaken idea that $\sqrt{2}$ is an "approximate" number or an "inexact" number. They should recognize that our work in this chapter really assures us that numbers like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ have a status precisely like that of the numbers with which they are more familiar.

In the first place, we can write indicated sums and products involving radicals, and we can decide when two expressions name the same number. That is, we can use our knowledge of the properties of the real numbers to show, for example, that

$$(\sqrt{2} - 3)(\sqrt{5} + 1) = \sqrt{10} - 3\sqrt{5} + \sqrt{2} - 3.$$

This means that we can work with irrational numbers in solving equations and in other instances.

Moreover, we can find rational approximations, to any accuracy, for a number such as $\sqrt{2}$. This means that if we wish to measure a length of $\sqrt{2}$ units with a meter stick or other device, we can do so as easily as we can measure a length of, say, $\frac{7}{9}$. Both measurements are accurate only within the limits of accuracy of our measuring device.

15-3. Simplification of Radicals

In proving $\sqrt{a}\sqrt{b} = \sqrt{ab}$, we make use of the fact that $\sqrt{ab}$ is unique. From this uniqueness it follows that if the square of a non-negative number is ab, then the number must be $\sqrt{ab}$, since there is only one non-negative number whose square is ab. You may wish to remind your class that similar logic was the basis of some of the earlier proofs we have had. It was used, for example, in showing that $\frac{1}{-a} = -\frac{1}{a}$, where we used the fact that a real number has only one multiplicative inverse.

The same observations apply to the proof that $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ in the next section.

15-4. Simplification of Radicals Involving Fractions

It is often convenient to rationalize the denominator of a fraction, particularly in numerical computations. Rationalizing the numerator is useful in later courses in calculating certain limits. Students should realize that the same number has many names. For example, $\frac{1}{\sqrt{3}}$ and $\frac{\sqrt{3}}{3}$ are simply two numerals for the same real number. Which we prefer depends on the use we are making of the number.

In connection with simplifying radicals, we give our usual connotation to "simplify": we perform as many indicated operations as we can.

15-5. Approximate Square Roots of Numbers Between 1 and 100
15-6. Approximate Square Roots of Positive Numbers

In Section 15-5 the student learns to approximate $\sqrt{a}$ to any pre-assigned degree of accuracy, if $1 < a < 100$. The iteration process by which this is done is discussed below. The iteration process could be used to approximate $\sqrt{a}$ for any non-negative a. However, it is somewhat simpler, if we are given a number which is not between 1 and 100, to write it first as the product of an even power of 10 and a number between 1 and 100. That is, every non-negative number a can be exp[illegible] as $10^{2n} \cdot b$ where n is an integer (hence, 2n is an even integer) an[illegible] $1 \leq b < 100$.

This fact can be used to keep the work with the iteration process as simple as possible. Moreover, the student needs to apply it when he uses a table of square roots. You may want to teach square root approximations from tables, if you have tables available.

Students should observe that the use of the iteration method in approximating square roots is not restricted to situations where we are trying to find a rational approximation to an irrational number. For example, students should notice Items 81 and 82 of Section 15-6. We can compute $\sqrt{1681}$ (Item 81) by factoring or by applying the iteration method. This is another opportunity to note that irrational numbers are handled, very often, like rational ones.

The Iteration Method of Approximating Square Root:

There is some controversy as to whether the square root algorithm or the iteration method of approximating square roots is superior. We advocate the latter for the following reasons.

1. The iteration method can be made meaningful more easily, since it is based directly on the definition of the square root: If $x^2 = n$, and $x \geq 0$, then $x = \sqrt{n}$. So the student must find a number which when squared gives n.
2. The student is estimating his results; thus, he is not likely to make a bad error without realizing it.
3. The second approximation can very often be done mentally, and always with very little arithmetic. In many cases it is all that is needed.
4. An easy division with a two-digit divisor yields a result in which the error is in the fourth digit. This is sufficient for most purposes.
5. The method is ideal for machine calculation.
6. The method is self-correcting.

Let us see how the process can be used to approximate $\sqrt{29}$. Let us call the approximations x_1, x_2, x_3, etc.

1. As the first approximation we take the closest integer, 5. Thus, $x_1 = 5$.
2. Average 5 and $\frac{29}{5}$ to find the second approximation. Thus,

$$\frac{1}{2}\left(5 + \frac{29}{5}\right) = 5.4.$$

3. Average 5.4 and $\frac{29}{5.4}$ for a third estimate. Thus,

$$\frac{1}{2}(5.4 + \frac{29}{5.4}) = 5.3852$$

$$x_3 = 5.3852.$$

(The approximation for $\sqrt{29}$ from a 5-place table is 5.38516.)

The iteration method thus yields a sequence of estimates x_1, x_2, x_3, etc., for $\sqrt{n}$. If x is any one of these estimates, then the next estimate is

$$\frac{1}{2}(x + \frac{n}{x}).$$

In order to explain why the iteration method works, it is helpful to use the following general theorem: If a, b are different positive real numbers, then

$$\frac{a + b}{2} > \sqrt{ab}.$$

Proof:

$$\left(\frac{a + b}{2}\right)^2 - (\sqrt{ab})^2 = \frac{a^2 + 2ab + b^2}{4} - ab$$

$$= \frac{a^2 - 2ab + b^2}{4}$$

$$= \frac{1}{4}(a - b)^2.$$

$\frac{1}{4}(a - b)^2$ is non-negative. Hence, it follows that

$$\left(\frac{a + b}{2}\right)^2 > (\sqrt{ab})^2.$$

Since $\frac{a + b}{2}$ and $\sqrt{ab}$ are non-negative, this inequality implies that

$$\frac{a + b}{2} > \sqrt{ab}.$$

($\frac{a + b}{2}$, the average of a and b, is often called the arithmetic mean of a and b. $\sqrt{ab}$ is called the geometric mean of ab. We can word our conclusion as: The arithmetic mean of two positive numbers is greater than their geometric mean.)

Let us apply this conclusion to the iteration process. Let x be positive. If $x \neq \sqrt{n}$, then one of the numbers x and $\frac{n}{x}$ is greater than $\sqrt{n}$ and the other is less than $\sqrt{n}$.

The iteration process involves taking an estimate x, and using it to form the new estimate $\frac{1}{2}(x + \frac{n}{x})$.

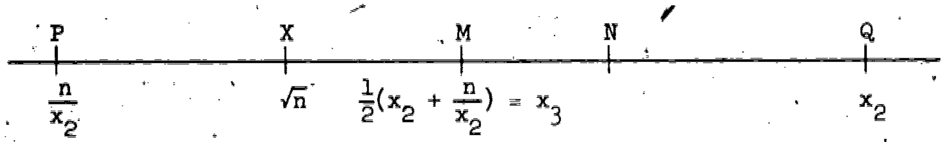
$\frac{1}{2}(x + \frac{n}{x})$ is the average (arithmetic mean) of x and $\frac{n}{x}$. The geometric mean of x and $\frac{n}{x}$ is $\sqrt{n \cdot \frac{n}{x}}$, or $\sqrt{n}$.

From the theorem above we conclude: Whatever estimate x we begin with

$$\frac{1}{2}(x + \frac{n}{x}) > \sqrt{n}.$$

That is, whatever estimate of $\sqrt{n}$ we begin with, the estimates yielded by the iteration process are certain to be greater than $\sqrt{n}$. (In the example we completed, x_2, x_3 are greater than $\sqrt{29}$, (although $x_1 < 29$).)

Suppose that we have an estimate x which is greater than $\sqrt{n}$. Let us take, for example, x_2. We compare x_2 with the average of x_2 and $\frac{n}{x_2}$; that is, with x_3.



$\frac{1}{2}(x_2 + \frac{n}{x_2})$, the average of x_2 and $\frac{n}{x_2}$, corresponds to the midpoint M of segment PQ. Note that N, the midpoint of XQ, is to the right of M. Hence, $XM < \frac{1}{2}XQ$. That is,

$$x_3 - \sqrt{n} < \frac{1}{2}(x_2 - \sqrt{n}).$$

Similarly, $$x_4 - \sqrt{n} < \frac{1}{2}(x_3 - \sqrt{n}),$$

or $$x_4 - \sqrt{n} < \frac{1}{4}(x_2 - \sqrt{n}).$$

Similarly, $$x_5 - \sqrt{n} < \frac{1}{8}(x_2 - \sqrt{n}).$$

We conclude: $x_2, x_3, x_4, \ldots$ become closer and closer to $\sqrt{n}$ from the right. A similar argument shows that $\frac{n}{x_2}, \frac{n}{x_3}, \frac{n}{x_4} \ldots$ become closer and closer to $\sqrt{n}$ from the left. That is, if we wish to find an estimate of $\sqrt{n}$ that differs from $\sqrt{n}$ by at most a given amount (say d), we can be sure that our process eventually will yield such an estimate.

Now we ask: How can we tell how accurate a given estimate is? In order to answer this, we compare $x_2 - \sqrt{n}$ (the length of XQ) and $\frac{1}{2}(x_2 + \frac{n}{x_2}) - \sqrt{n}$, (the length of XM).

$$x_3 - \sqrt{n} = \frac{1}{2}(x_2 + \frac{n}{x_2}) - \sqrt{n}$$

$$= \frac{x_2^{\,2} - 2\sqrt{n}\,x_2 + n}{2x_2}.$$

Thus, $$x_3 - \sqrt{n} = \frac{(x_2 - \sqrt{n})^2}{2x_2}. \qquad (1)$$

XM is the error of the approximation x_3, while XQ is the error of the preceding approximation, x_2. That is, the difference $x_2 - \sqrt{n} > x_2 - x_3$. Thus, we have from (1),

$$x_3 - \sqrt{n} < \frac{(x_2 - x_3)^2}{2x_2}.$$

Similarly, $$x_4 - \sqrt{n} < \frac{(x_3 - x_4)^2}{2x_2}, \text{ etc.}$$

Thus, in our example the error in our approximation x_3 is found:

$$x_3 - \sqrt{29} = 5.3852 - \sqrt{29}$$

$$5.3852 - \sqrt{29} < \frac{(5.4 - 5.3852)^2}{2(5.4)} \qquad (\text{since } x_2 = 5.4)$$

$$x_3 - \sqrt{29} \approx .00002.$$

You should notice that the formula

$$x_3 - \sqrt{n} < \frac{(x_3 - x_2)^2}{2x_2}$$

gives us the error in our approximation; that is, $x_3 - \sqrt{n}$, in terms of the numbers x_2 and x_3 which we have already found.

The above discussion was intended to help you understand why our iteration process works. In actually using the process, we follow some special rules about rounding off:

1. When we average x_1 and $\frac{n}{x_1}$ to find the second estimate x_2, we carry out the division indicated by $\frac{n}{x_1}$ to three digits and average the three digits.

2. In averaging x_2 and $\frac{n}{x_2}$ to obtain x_3, we round off x_2 to <u>two</u> digits before dividing, in $\frac{n}{x_2}$. We carry out the division to four digits and average to four digits. This estimate will usually exceed $\sqrt{n}$ by an error less than .002.

3. Should even more accuracy be required, round x_3 to three digits and then divide and average to six digits.

These rules give good results with a minimum of computation. (Rounding off may cause occasional slight discrepancies with the formulas derived above.)

You may be interested in observing the following table in which the third estimates for certain square roots are compared with values given by tables. In constructing this table we have purposely used instances of $\sqrt{n}$ for which the first estimate is not close. These are the cases in which we would expect the method to be least accurate. Notice that nevertheless the results are excellent.

	First Estimate	Second Estimate (1st. Ave.)	Third Estimate (2nd. Ave.)	From Tables
$\sqrt{2}$	1	1.50	1.41667	1.414214
$\sqrt{6}$	2	2.50	2.45000	2.449490
$\sqrt{13}$	4	3.64	3.60555	3.605551
$\sqrt{21}$	5	4.60	4.58261	4.582576
$\sqrt{30}$	5	5.50	5.47727	5.477226
$\sqrt{43}$	7	6.57	6.55757	6.557439
$\sqrt{57}$	8	7.56	7.55000	7.549834
$\sqrt{73}$	9	8.56	8.54418	8.544004
$\sqrt{91}$	10	9.55	9.53947	9.539392

Chapter 16

POLYNOMIALS AND FACTORING

This chapter and the next deal with polynomials, and the following general comments apply to both chapters. As to the mathematical content, you should be aware that there are two somewhat different ways of looking at polynomials in one variable. Let us consider, for example, the polynomial $x^2 - 5x + 6$. We may regard x as a variable in our usual sense; that is, as a definite number. $(x - 2)(x - 3) = x^2 - 5x + 6$ is then an open sentence which is true for all real values of x. We could verify this fact by a procedure that is by now quite familiar. The first step might be written:

If x is a real number, then $(x - 2)(x - 3) = x(x - 2)-2(x - 3)$

from the distributive property.

This point of view about the variable conforms to our earlier definition of variable. It conforms, also, to the logical requirements of one important outcome of this chapter, the solving of certain equations.

Let us consider, for example, the equation

$$x^2 - 5x + 6 = 0.$$

Since for any real number x,

$$x^2 - 5x + 6 = (x - 2)(x - 3),$$

we may say:

$x^2 - 5x + 6 = 0$ is true for some number x

if and only if $(x - 2)(x - 3) = 0$ is true for this same x.

From one point of view, then, we are staying within the familiar framework already developed for working with real numbers. On the other hand, it is possible to adopt the following somewhat different point of view. We consider a system whose elements have the form $a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$, where $a_0, a_1, \dots, a_n$ are real numbers. Examples of the elements of this system are:

$$\sqrt{2}x, \quad x^2 - 8x + 3, \quad x^3, \quad x + 1, \quad 7, \quad \text{etc.}$$

We do not regard x as a variable in our earlier sense. Rather, it is what is sometimes spoken of as an <u>indeterminate</u>--that is, a symbol with which we simply operate according to certain rules. The properties of this system are taken to be such that as long as one performs algebraic manipulations the results are exactly the same as those one obtains by regarding the variables as numbers.

In other words, in this mathematical system the statement

$$(x^2 - 5x + 6) = (x - 2)(x - 3)$$

is true, but it is now a statement that $x^2 - 5x + 6$ and $(x - 2)(x - 3)$ are two names for the same element of the abstract system.

What, then, is the advantage of this second approach? The important point is that when we consider x as an indeterminate, rather than a number, we are able to see easily that our new system of polynomials in one variable is similar in structure to the set of integers. Since we are familiar with the set of integers, this helps us understand better how certain ideas--such as factoring--relate to polynomials. It is closed, first of all, under addition, subtraction, and multiplication, but not division. Moreover, it possesses a unique factorization theorem and a division algorithm, both analogous to those which hold for the integers.

Experienced mathematicians are easily able to move back and forth between thinking of polynomials as elements in an abstract system and thinking of them as representing numbers. However, this ability to shift from one point of view to another requires more experience than beginning students have had.

The teacher--and textbook author--is confronted with a dilemma. On the one hand it is clearly unreasonable to attempt the formal introduction of a new, abstract mathematical system. Moreover, we want to encourage the student to test his work as he goes along by referring to things he already knows--namely, to properties of the real number system.

On the other hand, the analogies between the abstract system of polynomials over the real field and the set of integers are not only interesting but very helpful in understanding polynomials. We wish to avoid confusing the student with too many new and abstract ideas at once, while at the same time suggesting ideas he will use with increasing frequency in his later mathematical work.

The following suggestions as to teaching appear to be consistent with this over-all purpose and are used in this chapter.

1. Memorization of definitions should not be stressed. Although we need some vocabulary--polynomial, degree of a polynomial, and the like--the emphasis should be on simple recognition primarily. This is because our definitions have to be somewhat provisional.

2. In connection with each "type" of factoring situation, two steps are involved. First, a general pattern--for example,

$$a^2 - b^2 = (a + b)(a - b)$$

is developed from our knowledge of the properties of the real number system. Second, students are given suggestions as to how to recognize this pattern. Diagrams with arrow and similar devices are intended only to aid in this recognition. Students should be reminded frequently that they are only applying known ideas.

3. Where appropriate, the text notes that a given conclusion "reminds us", or "is similar to" one that we reached for the integers. We do not do much about this except note it, because we do not want to formalize the similarity. However, in discussions you may wish to be sure students do indeed remember the property of the integers in question.

4. Students should be encouraged to check their work, if they are in doubt, by using numbers.

5. We have noted (Section 16-1) that every polynomial can be expressed in "common polynomial form". Had we wished to adopt the abstract point of view noted, we would have regarded polynomials in common polynomial form as elements of our abstract mathematical system. We would have defined the operation in this system in such a way that the product of $(x - 2)$ and $(x - 3)$, for example, would be $x^2 - 5x + 6$, by definition. Our work in writing polynomials in common polynomial form is thus pointed toward a later, more abstract point of view, even though we do not make an issue of this.

6. As the student learns to factor new polynomials, he is given opportunities to apply his knowledge to solving equations. Solving equations thus to some extent provides motivation, and it also serves as a reminder of the relation between the work on factoring and our ideas about real numbers.

This chapter is very closely related to the next two. For this reason, it is important to have in mind what degree of mastery is appropriate at certain stages.

16-1. Polynomials

This section deals with addition and subtraction of polynomials. The material in subtraction may seem inadequate. It will be reinforced, however, in Chapter 18, where more drill is provided. This arrangement seems appropriate, because we use subtraction, of course, in connection with division of polynomials in Chapter 18. Therefore, additional drill material is not likely to be needed in Section 16-1, where we are mainly interested in stressing general ideas about polynomials.

16-2. Factoring by the Distributive Property

When we write an expression of the form $ab + ac$ as the product $a(b+c)$, we are simply applying the distributive property. We emphasize this point. In order to apply the distributive property, we must first be able to recognize when a given polynomial has the form $ab + ac$. Thus, we must be able to identify a. Now a is simply a common factor of the terms, and hence our ideas of common factor, and of greatest common factor, are relevant.

Beginning with Section 16-5 we will be increasingly focusing on quadratic polynomials. The material on factoring by grouping terms will not be needed. Students should understand the idea of factoring by grouping, but they will vary greatly in their ability to select the terms to group.

16-3. Difference of Squares

16-4. Perfect Squares

16-5. Factoring by Completing the Square

These three sections are closely related. The ability to recognize perfect squares (Section 16-3), and to factor the difference of two squares (Section 16-4), are both needed in factoring by completing the square.

Completing the square will be used over and over again in Chapter 17, and consequently, it is not expected that students will have developed a great deal of skill in it at the end of Chapter 16. It is very important, however, that they understand thoroughly the idea involved.

16-6. Summary and Review

Chapter 17

QUADRATIC POLYNOMIALS

17-1. Factoring by Inspection

17-2. Factoring by Inspection, (continued)

The student should understand that polynomials which can be factored over the integers by inspection can also be factored by the method of completing the square. They should become familiar with both methods.

In factoring polynomials over the integers, intuition and ingenuity play a role. Students should not be discouraged from guessing, nor should they be required to follow a rigid sequence of steps in factoring. The very process of testing a wrong guess may help a student to greater understanding.

Some students will enjoy using their knowledge about factoring integers to find short cuts in factoring polynomials.

17-3. Factoring Over the Real Numbers

Between the consideration of factoring polynomials over the integers and that of factoring polynomials over the real numbers, it would have been possible to give more attention to factoring polynomials over the rational numbers.

Suppose, for example, we wish to write $\frac{1}{4}x^2 - \frac{5}{4}x + 1$ as a product of polynomials of lower degree with rational coefficients. The process is extremely simple. We have:

$$\frac{1}{4}x^2 - \frac{5}{4}x + 1 = \frac{1}{4}(x^2 - 5x + 4)$$
$$= \frac{1}{4}(x - 4)(x - 1).$$

Indeed, we can always write any polynomial over the rationals as the product of a rational number ($\frac{1}{4}$ in the example) and a polynomial with integer coefficients. This fact is easy to prove.

Suppose we apply this process to the polynomial $\frac{1}{3}x^2 + x + \frac{1}{3}$:

$$\frac{1}{3}x^2 + x + \frac{1}{3} = \frac{1}{3}(x^2 + 3x + 1).$$

$x^2 + 3x + 1$ is not factorable over the integers. It is not at all obvious that it consequently is also not factorable over the rational numbers. The more numbers you have, so to speak, to build polynomials with, the better chance of a factorization. However, it is a fact, which follows from a lemma of Gauss, that if a polynomial over the integers is not factorable over the integers, then it also is not factorable over the rational numbers.

On the other hand, $x^2 + 3x + 1$ __is__ factorable over the real numbers:

$$x^2 + 3x + 1 = (x + \frac{3}{2})^2 - \frac{5}{4}$$

$$= (x + \frac{3 + \sqrt{5}}{2})(x + \frac{3 - \sqrt{5}}{2}).$$

Incidentally, one immediate consequence of Gauss' Lemma is the irrationality of $\sqrt{2}$. For if $\sqrt{2}$ were rational, then $x^2 - 2$, or $x^2 - (\sqrt{2})^2$, would be factorable over the rationals. But $x^2 - 2$ is a polynomial which is factorable over the integers. Hence, it cannot be factorable over the rationals.

As we see, factoring polynomials over the rationals does not give us any new factoring possibilities of importance. We have chosen to minimize any discussion of it for two pedagogic reasons. First, the student will need factorizations like $\frac{1}{3}(x - \frac{1}{2})^2$ in connection with his study of parabolas. Second, the manipulative problems of factoring are time consuming, and we have felt it wise to minimize extraneous matters.

17-4. Quadratic Equations

Writing the standard form of the quadratic polynomial is an easy exercise in completing the square. It will be a useful tool in dealing later with graphs of parabolas.

The quadratic formula concludes this chapter. It is inserted as a natural generalization of the solution of quadratic equations by completing the square. It is __not__ intended that the student acquire mastery of its use. The wide applicability of the technique of completing the square makes its continued use in solving quadratic equations justifiable. We want the student to continue to improve his skill in using it. Moreover, the interpretation of the quadratic formula will be clearer, and its use consequently less mechanical, when the student can relate it to the graph of the quadratic polynomial, which will be discussed in Chapter 23.

17-5. Summary and Review

INTRODUCTION TO PART 4

(Chapters 18-24)

This volume contains material on a variety of topics which arise naturally from the material of Parts 1-3. Chapter 18 deals with the division algorithm for polynomials and with the algebra of rational expressions. The development points out the analogies between integers and rational numbers on the one hand, and polynomials and rational expressions on the other.

Chapter 19 is devoted to a thorough treatment of the matter of equivalent open sentences. [illegible] interesting and challenging questions regarding the reversibility of steps in obtaining a chain of equivalent sentences arise in connection with sentences involving rational expressions. Inequalities are treated [illegible] with equations. The non-reversibility of "squaring both sides" is [illegible].

In Chapter [illegible] introduce equations in two variables. Since the truth set of [illegible] in two variables is a set of <u>ordered pairs</u>, we are led [illegible] of the real number <u>plane</u>, and to the graph of the truth set of an equation in two variables. This leads into the systematic development of the equation of the line in Chapter 21.

In Chapter [illegible] of two equations of first degree in two variables are treated. The [illegible] process of solving such systems is related to the [illegible].

The [illegible] polynomials are developed in Chapter 23. In the course of the [illegible] opportunity for the student to apply his earlier experience with quadratic polynomials.

[illegible] introduces the concept of function and the standard [illegible]. The relation between the function concept and the graphs of open sentences in two variables is stressed. The student learns to restate his [illegible] parabolas in terms of functions.

[illegible], the student [illegible] his earlier knowledge in a variety of ways. We hope that he [illegible] aware of the way in which one mathemat- [illegible]. [illegible], in one sense he should regard this material [illegible]. In another sense, however, we are laying the [illegible] developed during the student's further mathe- [illegible]. Every [illegible] of mathematical learning represents a new [illegible], and [illegible] for the student.

Chapter 18

DIVIDING POLYNOMIALS; RATIONAL EXPRESSIONS

18-1. Division of Polynomials

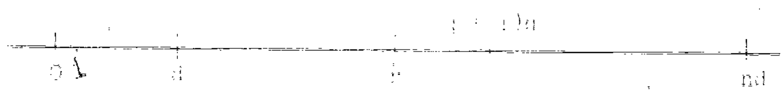
18-2. Division of Polynomials, Concluded

The already noted parallelism in structure between the set of integers and the set of polynomials in one variable is extended further in Sections 18-1 and 18-2. These sections deal with the division algorithm.

We review the division algorithm for positive integers:

> For any positive integers n and d, there are integers q and r such that
>
> $$n = qd + r, \qquad 0 \le r < d.$$

The proof of this can be described intuitively by using the number line. First, if $d > n$, then $n = 0 \cdot d + n$, and $n < d$. If $d = n$, then the remainder is 0.



If d and n are positive integers with [illegible], then some integer multiple of d is certainly greater than n. For example, since $d \ge 1$, $nd \ge n$. Of the set of integer multiples of d which are greater than n, choose the smallest, and call it $(q + 1)d$. We have:

$$qd \le n < (q + 1)d$$
$$0 \le n - qd < d.$$

Setting $n - qd$ equal to r, we have:

$$n = qd + r, \qquad 0 \le r < d.$$

Though [illegible], [illegible] the fact that the set of integer multiples of [illegible] is not an immediate consequence of the properties of [illegible]. It follows [illegible] stated in these Notes [illegible] the so-called Well-ordering Principle, which states: [illegible] set of positive integers contains a smallest element.

The set of [illegible] also has a division algorithm: If [illegible], then there are polynomials Q, R such that and either R is 0 or the degree of R is less than that of D. (Note the comparison: For integers, we have $0 \leq r < d$. For polynomials, we have either the degree of R is less than the degree of D or R is 0.)

The proof of the division algorithm for the case in which N has degree 2 and D has degree 1 is very simple.

$$\text{If} \qquad N = ax^2 + bx + c$$
$$D = ex + f, \qquad (e \neq 0).$$

We need only apply the long division process described in the student text. That is, we first subtract a suitable multiple of D; namely $\frac{a}{e}D$, to eliminate the x^2 term and then another multiple to eliminate the x term.

A general proof of the division algorithm, which would hold for all polynomials N and D, would use the same idea but would require mathematical induction.

It is interesting to observe that the division algorithm holds only if we are dealing with a set of polynomials for which every coefficient except 0 has an inverse. Hence, the division algorithm does not hold if we limit ourselves to polynomials over the integers. (The student who completes Items *44 to *4[illegible], Section 13-2, might like to note this fact.)

13-3. Products and Quotients Involving Polynomials.

13-4. Rational Expressions

These sections deal with rational expressions. We have noted already (Teachers' Notes, Chapter [illegible]) that statements about polynomials can be interpreted from two points of view. For example,

$$x(x + 1) = x^2 + x$$

may be [illegible] the domain of the variable in the set of real numbers. However, we [illegible] regard it as a statement that $x(x + 1)$ and $x^2 + x$ name the same element in a certain abstract mathematical system formed from the real numbers and the indeterminate x.

A similar [illegible] in connection with, for example,

$$[illegible]$$

Here, however, we [illegible] that needs noting. If we regard

$$[illegible]$$

[illegible] of real numbers, then

we must exclude 0 from the domain. We do not have a true sentence if x is 0. However, if we regard x as an indeterminate, then $\frac{1}{x} + \frac{1}{x^2} = \frac{x + 1}{x^2}$ is again a statement about elements in our abstract system, and remarks about the domain of x would be meaningless.

We feel that to adopt the second point of view here would be confusing, given the assumed level of maturity of the students. Hence, we have emphasized the first point of view and with it the need to exclude certain values of the variable in dealing with rational expressions. In this way, we are using an approach which "spirals" into Chapter 19, where "fractional" equations are discussed and where, consequently, values of the variable which would lead to division by 0 must be treated with great care.

13-5. Summary and Review

Chapter 19

TRUTH SETS OF OPEN SENTENCES

19-1. Equivalent Equations

The student has had a great deal of experience in solving equations by constructing a chain of equivalent equations. The objective, of course, is to obtain an equation which has an obvious truth set. At the beginning of this section we review how one equation is equivalent to a second if one is obtained from the other by a "reversible step". The permissible steps are:

1. Adding (or subtracting) a real number to both sides.

2. Multiplying (or dividing) both sides by a non-zero real number.

Difficulties arise if we wish to add or subtract a phrase containing a variable. Thus, $x = 3$ is equivalent to, say, $(2x - 1) + x = (2x - 1) + 3$, since $2x - 1$ names a real number no matter what number x represents. On the other hand, $x = 3$ is not equivalent to $x + \frac{1}{x} = 3 + \frac{1}{x}$, since, for one value of x, $\frac{1}{x}$ does not name a real number. Notice, however, that $x = 3$ is equivalent to the compound sentence "$x + \frac{1}{x} = 3 + \frac{1}{x}$ and $x \neq 0$".

A more usual case arises in trying to solve, for example, $x(x-2) = 3(x-2)$. Dividing by $x - 2$ yields $x = 3$, which is not equivalent to the given equation, as may be seen by comparing the truth sets. We might solve $x(x-2) = 3(x-2)$ as follows:

$$x(x - 2) = 3(x - 2)$$

$$x(x - 2) - 3(x - 2) = 0 \qquad \text{, subtracting } 3(x - 2)$$

$$(x - 3)(x - 2) = 0 \qquad \text{, distributive property}$$

$$x - 3 = 0 \text{ or } x - 2 = 0 \qquad \text{, } ab = 0 \text{ implies } a = 0 \text{ or } b = 0.$$

Notice that each of the steps taken above is reversible.

In the text we point out that since if $ac = bc$, then $a = b$ or $c = 0$. This used to the student to write in one step that

$$x(x - 2) = 3(x - 2) \quad \text{is equivalent to}$$

$$x = 3 \text{ or } x = 2.$$

These, and similar matters, are treated in this section.

19-2. Equations Involving Fractions

We now apply the ideas of the preceding section to equations such as

$$\frac{1}{y} - \frac{1}{y - 4} = 1.$$

It should be observed that when we write a rational expression, we agree that the domain of the variable must exclude values that make any denominator 0. Thus,

$$\frac{1}{y} - \frac{1}{y - 4} = 1 \quad \text{is equivalent to}$$

$$\frac{1}{y} - \frac{1}{y - 4} = 1 \quad \text{and} \quad y \neq 0 \quad \text{and} \quad y \neq 4.$$

19-3. Squaring Both Sides of an Equation

We begin with a word problem which leads to the open sentence

$$\sqrt{8^2 + n^2} = n + 4.$$

It is natural to approach this equation by "squaring both sides". In this case, the solution is obtained without difficulty. We then deal with situations in which squaring both sides of an equation does not lead to an equivalent equation. In fact, since

$$\text{if } a = b, \quad \text{then } a^2 = b^2,$$

it follows that the solution set of a given equation is a subset of the solution set of the equation obtained by squaring both sides.

The truth set of $x = 3$ is a subset of the truth set of $x^2 = 9$. Those solutions of the equation obtained by squaring, which are not solutions of the original equation, have traditionally been called "extraneous solutions" of the original equation. We avoid this terminology. For us, a solution is a member of a truth set. Any number which is not in the truth set is not a solution.

Since "squaring" does not lead to a reversible step, the student is cautioned that checking is absolutely necessary whenever the squaring technique is used.

The student knows that $\sqrt{x^2} = |x|$ for all real numbers x, so he should be able to see that equations involving absolute value may also be attacked by squaring.

We remind the student that the domain of the variable may have to be restricted. Thus, $\sqrt{x - 1}$ does not name a real number if x is less than 1.

19-4. Inequalities

The major tools to be used in finding the solution set of an inequality are the addition and multiplication properties of order. The addition property of order presents no essentially new problem. On the other hand, the multiplication property of order must be used with caution.

We first review the material on solving inequalities which has been presented earlier in the program, particularly in Chapter 10.

The problem of multiplying both sides of an equality by a phrase which is either positive for all values of the variable or is negative for all values is discussed briefly. That is, $\frac{x}{x^2 + 2} < 0$ is equivalent to $x < 0$, since $x^2 + 2$ is positive for all real numbers x. Similarly, $\frac{x}{-x^2 - 2} < 0$ is equivalent to $x > 0$, since $-x^2 - 2 < 0$ for all real numbers x.

If a phrase is positive for some values of the variable, zero for other values, and negative for still others, then we must investigate all three possibilities. Suppose $a < 2$ is true. If both sides of this inequality are multiplied by a, what conclusion may be drawn about the order of a^2 and $2a$?

If $0 < a < 2$, then $a^2 < 2a$.

If $a = 0$, then $a^2 = 2a$.

If $a < 0$, then $a^2 > 2a$.

We discuss this question in connection with inequalities such as $(x - 1)(x - $ [illegible]$) > 0$ which arise in the study of elementary calculus. This portion of the section is starred, but it is not extremely difficult.

19-5. Summary and Review

Chapter 20

THE GRAPH OF $Ax + By + C = 0$

In this chapter we begin the study of open sentences in two variables and the graphs of these sentences. We motivate the use of ordered pairs as coordinates of points in the plane by the need of ordered pairs to denote the elements of the truth set of an open sentence in two variables and concentrate on the equation $Ax + By + C = 0$ with not both A and B zero.

20-1. The Real Number Plane

We introduce the use of ordered pairs for solutions of the equation $Ax + By + C = 0$ and we show that there is a one-to-one correspondence between the set of all ordered pairs of real numbers and the points of the plane. The coordinate axes are introduced and the student is given practice in locating points with given coordinates and finding the coordinates of a given point. Much of this will be familiar to the student and he may be able to do this section rather quickly.

20-2. The y-Form of the Equation of a Line

We begin with the general equation, $Ax + By + C = 0$ with not both A and B zero. The student is led to discover that all the solutions of this equation are on a line and that all the points on this line have coordinates which satisfy this equation. The particular equations: $x = a$ and $y = b$ may also be considered as open sentences in two variables; that is, the equation $x = a$ can be written $x + 0y - a = 0$ and the equation $y = b$ can be written $0x + y - b = 0$. The graph of $x = a$ is a vertical line and the graph of $y = b$ is a horizontal line.

The y-form of the equation $Ax + By + C = 0$ facilitates graphing since the ordinate (y-value) of each point of the graph may be found in terms of the abscissa (x-value). This is, of course, the slope-intercept form of the equation of a line which the student will encounter in analytic geometry.

20-3. Definition of Slope and y-Intercept

We begin this section with the construction of the graphs of the equation $y = kx$ for various values of k. [illegible] From them the student discovers that the direction of the line depends on the coefficient of x in the y-form of the equation.

In the next sequence of items, we construct the graphs of the equations $y = \frac{2}{3}x + b$ for various values of b. Here the student finds that the lines all have the same direction, but the intersection of the line with the y-axis depends on th[illegible] value of b.

We are [illegible] to the definition of slope and y-intercept of a line. We distinguish between the y-intercept number, the number b in the equation $y = mx + b$, and the y-intercept which is the coordinate of the point of intersection of the line and the y-axis; i.e., the point $(0,b)$.

We have, essentially, a choice between two possible definitions of the slope of a line: the coefficient of x in the y-form of the line; the ratio of the vertical change to the horizontal change from one point to another on the (non-vertical) line. In a course in analytic geometry, in which a line is given a geometric meaning, the second of these would be taken as definition and the first proved as a theorem. Here we have really defined a line in terms of its equation, and it is natural to take the first as a definition and prove the second.

We prefer to say that the slope of a vertical line is "undefined" rather than to say that a vertical line has no slope. This is in keeping with usual terminology and also should serve to lessen the confusion between this idea and the fact that a horizontal line has zero slope.

10-4. Applications of Slope and Intercept

In this section we show in some examples that the ratio of vertical change to horizontal change from one point to another on the line is the same as the slope of the line. This is stated as a theorem and the proof is given in starred items.

The student is given practice in graphing the line given the slope and intercept, the slope and any one point, two points, etc.

The "two-point form" of the equation of a line is not derived, although we write the equation of the line in y-form or standard form, given the coordinates of two points.

For example, given the points [illegible] and [illegible], what is the equation of the line? We find that the slope is [illegible] and if we suppose (x,y) are coordinates of a third point, we may write

[illegible]

However, if we think of [illegible] the point [illegible] is not in the domain of definition [illegible]

$$y = -x - 1$$

which is satisfied by all the points of the line through the given points. However, the two equations are not equivalent since they do not have the same truth set.

20-5. Summary and Review

Chapter 21

GRAPHS OF OTHER OPEN SENTENCES IN TWO VARIABLES

21-1. Graphs of Inequalities

We compare the ordinate of a point on a line $y = 3x$ with the ordinates of points with the same abscissa above this line. The line divides the plane with two half-planes. The region above the line is the set of all points satisfying the open sentence $y > 3x$. If the points of the line are included in the region, the corresponding open sentence is $y \geq 3x$. Thus, we are able to graph open sentences in two variables which are inequalities as well as equalities.

This section gives the student practice in graphing these open sentences. Although it is suggested that the student find a form of the inequality like the y-form of an equation of the line, the student should not be discouraged from other approaches.

21-2. Graphs of Open Sentences Involving Absolute Value

This section dealing with absolute value is valuable not only for the opportunity it provides for recall of work done with absolute value earlier in the course, but also for the opportunity it provides for examining what happens to a graph when certain changes are made in its equation.

The simplest case is an open sentence such as $|x| = 3$. This is now considered as an open sentence in two variables and the truth set is the union of the truth sets of the sentences $x = 3$ and $x = -3$. Since each of these is the equation of a line, the graph of $|x| = 3$ is two lines.

The graph of $|x| < 3$ may be described in two ways. We may consider the sentence $-3 < x < 3$ which is the same as "$x < 3$ and $x > -3$". The truth set of this sentence is the intersection of the truth sets of $x < 3$ and $x > -3$.

This is also equivalent to the sentence "$x \geq 0$ and $x < 3$ or $x < 0$ and $x > -3$". Using the latter sentence allows us to consider the truth set of $|x| < 3$ as the union of two truth sets, parallel to the case $|x| > 3$. Either of these descriptions is correct.

In the second part of this section we graph the equation $y = |x|$. The student will meet this graph many times in his study of mathematics. Do not insist on complete mastery at this time.

*21-3. <u>Graphs of Open Sentences Involving Integers Only</u>

This section is included because it is hoped the student will realize that open sentences do not necessarily include all real numbers as possible members of their truth sets, and will recognize the corresponding situation so far as the graphs are concerned. The section is starred in this case because it is out of the mainstream of the course, not because of any difficulty in the ideas.

In Section 22-4 an example is given for which the solution sets consist of ordered pairs of positive integers. This will present no difficulty to the student, even if he has not read this section.

21-4. <u>Summary and Review</u>

The summary of the material is given in tabular form for convenience. The student should not memorize this table.

Chapter 22

SYSTEMS OF EQUATIONS AND INEQUALITIES

22-1. Systems of Equations

22-2. Systems of Equations (Continued)

The system of linear equations

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0, \end{cases}$$

that is, the conjunction

$$Ax + By + C = 0 \ \underline{\text{and}} \ Dx + Ey + F = 0,$$

arises in many contexts where two variables have two conditions placed on them simultaneously. This probably explains why such a system is often called a "system of simultaneous equations".

We want the students to continue extending their ideas about sentences, truth sets, and graphs. Thus, such a system is another example of a sentence in two variables, and we again face the problem of describing its truth set and drawing its graph. As before, we solve this sentence by obtaining an equivalent sentence whose truth set is obvious. Here we are aided by the intuitive geometry of lines. Two lines either intersect in exactly one point or they are parallel. If the lines given by the system intersect, then the point of intersection must have coordinates satisfying both equations of the system, and this ordered pair is the solution of the sentence. Thus, the problem is one of finding two lines through this point of intersection whose equations are the most simple; namely, a vertical line and a horizontal line. All methods of solving such systems are actually procedures for finding these two lines.

Our purpose here is to derive a method for solving a system of equations

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases} \qquad (A \neq 0 \ \text{ or } \ B \neq 0; \ \ D \neq 0 \ \text{ or } \ E \neq 0.)$$

on the assumption that this system has exactly one ordered pair of real numbers as its solution. We think of the individual clauses of the system as equations of lines and try to find a vertical line and a horizontal line which pass through the point common to this first pair of lines. If these new lines have equations "$x = a$" and "$y = b$", then (a,b) is the solution of our system.

The lines of the system $\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases}$

have exactly one point in common if and only if they are neither parallel nor coincident. We know that two lines are parallel or coincident if and only if they are both vertical or both have the same slope. Putting these statements together we can say that the lines of the above system have <u>exactly one point in common if and only if</u>:

1. $B \neq 0$ and $E \neq 0$ (neither line is vertical), and $-\frac{A}{B} \neq -\frac{D}{E}$ (the lines are not parallel or identical); <u>or</u>
2. One of the numbers B and E is 0 and the other is not (one line is vertical and the other is not).

A very acute student may inquire how these conditions affect the method for solving linear systems which is given in the text. He may wonder why, for example, we are <u>always</u> able to select proper multipliers r and s which yield equations of horizontal and vertical lines through the point common to the given lines, if there <u>is</u> exactly one point. The relation between the algebraic solution process and the geometry depends on the following theorem.

<u>Theorem</u>. The lines of the system

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases} \quad \begin{array}{l} \text{(where } A, B \text{ are not both } 0 \\ \text{and } D, E \text{ are not both } 0.) \end{array}$$

are parallel or coincident if and only if there exist real numbers r and s, not both zero, such that

$$rA + sD = 0 \quad \text{and} \quad rB + sE = 0.$$

<u>Proof</u>: Let us prove first that if the lines are parallel or coincident, then there exist numbers r and s, not <u>both</u> zero, such that $rA + sD = 0$ and $rB + sE = 0$. There are two cases. Either the lines are vertical, or they have the same slope. If they are vertical, then

$$B = 0, \quad E = 0, \quad A \neq 0, \quad D \neq 0,$$

and we may choose $r = -D$ and $s = A$ so that

$$rA + sD = (-D)A + AD = 0 \quad \text{and} \quad rB + sE = r\cdot 0 + s\cdot 0 = 0.$$

If the lines have the same slope, then $-\frac{A}{B} = -\frac{D}{E}$; that is, $\frac{A}{B} = \frac{D}{E}$. ($B \neq 0$ and $E \neq 0$.) In this case we may choose $r = -E$, $s = B$.

Then $rA + sD = (-E)A + BD = 0$ from the equality of $\frac{A}{B}$ and $\frac{D}{E}$.

$$rB + sE = (-E)B + B(E) = 0.$$

Conversely, let us assume that there exist real numbers r and s, not both zero, such that $rA + sD = 0$ and $rB + sE = 0$. We need to show that the lines are parallel or coincident. Since r and s are not both 0, let us suppose $r \neq 0$.

If an ordered pair satisfies

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases}$$

then it satisfies

$$r(Ax + By + C) + s(Dx + Ey + F) = 0. \qquad (1)$$

Since $rA + sD = 0$ and $rB + sE = 0$, by our assumption, then equation (1) becomes

$$rC + sF = 0. \qquad (2)$$

If (2) is false, then no ordered pair satisfies the system; that is, the lines are parallel.

If (2) is true, then since $r \neq 0$ we have

$$-\frac{s}{r}D = A$$

$$-\frac{s}{r}E = B$$

$$-\frac{s}{r}F = C.$$

That is, $Ax + By + C = 0$ and $Dx + Ey + F = 0$ are equivalent equations, and every point which lies on one line lies on the other as well. In this case, the two lines coincide.

This theorem allows us to justify the method of the text. Suppose we wish to solve

$$Ax + By + C = 0$$
$$Dx + Ey + F = 0$$

where A and B are not both 0 and C and D are not both 0, and where the two lines have exactly one point in common. Let us suppose $A \neq 0$ or $D \neq 0$. We take $r = D$, $s = -A$.

$$D(Ax + By + C) - A(Dx + Ey + F) = 0$$

is an equation in which the coefficient of x is clearly 0. However, the coefficient of y is not 0, since if $DB - AE = 0$ we would have both $rA + sD = 0$ and $rB + sE = 0$ in violation of our theorem.

Similarly, if $B \neq 0$ or $E \neq 0$, we can obtain an equation in which the coefficient of y is 0 but the coefficient of x is not 0.

The cases where A and D are both 0, or E and F are both 0, can easily be seen to involve parallel lines.

We do not expect students to follow an exposition of this length. However, we hope they will understand that when we have two equations for which the lines have one and only one point in common, then we can write an equivalent system for which the solution is obvious.

The use of the form $r(Ax + By + C) + s(Dx + Ey + F) = 0$ emphasizes the idea of what is sometimes called the "addition" method, or the "elimination" method. Once students have understood this idea, they are encouraged to use the more customary form in writing up their solutions.

In Sections 22-1 and 22-2 we concentrate on the case where the given lines are neither parallel nor coincident.

22-3. Parallel and Coincident Lines; Solution by Substitution

We discuss systems of the form

$$\begin{cases} Ax + By + C = 0 \\ kAx + kBy + F = 0 \end{cases} \qquad B \neq 0,\ k \neq 0,\ C \neq F,\ C \neq 0 \text{ or } F \neq 0$$

It is pointed out that the lines of this system are parallel, since they have the same slope but have different y-intercepts. The student is led to see that the methods of the preceding sections, if applied to this system, will result in the (false) numerical sentence $kC - F = 0$ ($k \neq 0$, $C \neq 0$ or $F \neq 0$). On the other hand, if the lines had some point in common, then $kC - F = 0$ would have to be true. Thus, both geometric considerations and our algebraic techniques enable us to see that the truth set of the system is $\emptyset$.

Similarly, the graph of a system of the form

$$\begin{cases} Ax + By + C = 0 \\ kAx + kBy + kC = 0 \end{cases} \qquad B \neq 0,\ k \neq 0,\ C \neq 0$$

consists of a single line (the "two" lines have the same slope and same y-intercept). The algebraic procedures of Section 22-1 and 22-2 lead to the (true) numerical sentence $0 = 0$.

In order to decide, before attempting to solve the system, whether two given lines are parallel or coincident, it is convenient to find the slope of each. This idea leads us to write each equation in y-form.

$$\begin{cases} y = m_1 x + b_1 \\ y = m_2 x + b_2 \end{cases}$$

If $m_1 = m_2$ and $b_1 \neq b_2$, the lines are parallel. If $m_1 = m_2$ and $b_1 = b_2$, the lines are coincident. If $m_1 \neq m_2$, then the lines have a single point of intesection (c,d). In this latter case $d = m_1 c + b_1$ and

$d = m_2c + b_2$, are both true sentences. To determine c, we argue that $m_1c + b_1$ and $m_2c + b_2$ are both names for d; hence, $m_1c + b_1 = m_2c + b_2$ and we may solve this equation for c. This discussion leads, then, in a natural way, to the solution of a system of linear equations by the "substitution" method.

Obviously, the argument above fails if one of the lines is vertical and therefore has no y-form. This special case is discussed in Items 63-71. Finally, we point out that, even if neither line is vertical, it is sometimes convenient to solve one or both equations for x rather than for y.

22-4. Systems of Inequalities

We have, throughout this course, treated inequalities along with equations. It is natural, then, to examine systems of linear inequalities in this chapter. Our technique of solving such a system is, of course, graphical. The material of this section provides a sound elementary introduction to linear programing and to the study of convex sets.

22-5. Review

Chapter 23

GRAPHS OF QUADRATIC POLYNOMIALS

23-1. Graphs of Equations of the Form $y = ax^2 + k$

23-2. Graphs of Equations of the Form $y = a(x - h)^2 + k$

This chapter deals with graphing polynomials of the form $ax^2 + bx + c$. The algebraic techniques for working with such polynomials have been developed in Chapter 17. Students who need review at this point should be referred to the relevant portions of Chapter 17.

Students who have mastered the process of completing the square should find it easy to rewrite the equation $y = ax^2 + bx + c$ in the equivalent form $y = a(x - h)^2 + k$. They should also be able to recognize that the graph of this equation represents a translation of the equation $y = ax^2$. In order to obtain the graph of $y = a(x - h)^2 + k$, we move the graph of $y = ax^2$

$|h|$ units to the right if h is positive; or
$|h|$ units to the left if h is negative;

and we move it

$|k|$ units up if k is positive;
$|k|$ units down if k is negative.

Little would be gained by requiring memorization of this rule. In later courses the student will become familiar with the concept of directed distance, which will enable him to formulate the procedure more easily, and he will study in more detail the idea of translation. Faithful to the idea of the spiral curriculum, we feel it appropriate to limit the objectives here to:

1. Some intuitive feeling for the situation.

2. A recognition that the idea of "moving" one graph to obtain another carries with it the explicit idea of a particular correspondence (actually, a one-to-one correspondence) between the points of one graph and those of the other.

In actually graphing $y = a(x - h)^2 + k$, we would prefer that the student locate the vertex by using its property of being the "highest" point on the curve (if $a < 0$) or the "lowest" (if $a > 0$).

This method exemplifies a major objective--that of gaining some understanding of how an algebraic property of an equation is related to a geometric observation about a curve.

23-3. Quadratic Polynomials of the Form $y = ax^2 + bx + c$

The student has already solved quadratic equations by completing the square. Some reteaching and reinforcement may be needed here. The relationship between an algebraic statement--the solution set of $ax^2 + bx + c$ is $\emptyset$--and a geometric one--the parabola $y = ax^2 + bx + c$ does not intersect the x-axis--supplies another instance of the point of view we want to stress.

This section uses a number of ideas that have been developed earlier. The teacher who is pressed for time may wish to work through a few selected examples as a cumulative review and as an illustration of the way in which several ideas can be combined. If a choice must be made, it would be better to leave this chapter before mastery is gained than to omit the next one.

23-4. Summary and Review

Chapter 24

FUNCTIONS

The concept of functions is basic to mathematics, in much the same way that the set concept is basic. It is implicit in many things the student has already done, not only in mathematics but in other subjects. He probably has seen in social studies, for example, tables associating with each year the population of a certain city for that year.

24-1. The Function Concept

We say that we have a function when we have a rule for associating with each number of a certain set of numbers (the domain) exactly one number of a second set (the range).

According to this definition, the domain and range are sets of real numbers, since a "number", for purposes of this course, is a real number. In later courses the student will meet functions in which the domain and range are sets other than real numbers. Such a function might, for example, have sets of points in the plane as its domain of definition. As an illustration, associate with each point (x,y) of the plane the abscissa x of the point. In this case the domain is the set of all points of the plane and the range is the set of all real numbers. If we associate with each point (x,y) of the plane, the point $(-x,y)$, the result is a function with both domain and range the set of all points in the plane.

In the discussion about functions, it is important to emphasize at every opportunity the following points:

1. To each number in the domain of definition, the function assigns one and only one number from the range. In other words, we do not have "multiple-valued" functions. However, the same number of the range can be assigned to many different elements of the domain.

2. The essential idea of function is found in the actual association from numbers in the domain to numbers in the range and not in the particular way in which the association happens to be described.

3. Not all functions can be represented by algebraic expressions.

24-2. The Function Notation

The examples in this section provide opportunities for emphasizing further the three points listed above.

Students should understand that in particular the same function can be described in many ways. Two descriptions lead to the same function whenever (1) the same set of numbers is the domain of both; (2) with each element of the domain the same number is associated. Notice that two functions are different when they have different domains, even though the same algebraic sentence may be involved in the description of both.

24-3. Graph of Functions

Let us consider a set of ordered pairs, such that no two elements of the set have the same first number. Such a set of ordered pairs describes a function. The first numbers which occur in the pairs of the set are the elements of the domain of the function. For ~~an ordered~~ pair (a,b) in our set, a is an element of the domain, and b is the number associated with a. Notice that in order that our set of ordered pairs describe a function, it is essential that it not contain two pairs with the same first number. We would not have a function if two different numbers were associated with the same element of the domain.

On the other hand, if we have a function, then we can always consider the set of ordered pairs (a,b) in which a is an element of the domain and b the number associated with a by the function.

Since every set of ordered pairs corresponds to a set of points, every function has a graph; namely, the graph of the set of ordered pairs.

Notice particularly that the graph of a function cannot contain two different points with the same abscissa.

The relation between functions and certain kinds of sets of ordered pairs is very close, as we see. Indeed, we might have defined a function as a set of ordered pairs of numbers such that no two elements have the same first number.

24-4. Linear and Quadratic Functions

The student is likely to encounter the terms "direct variation" and "inverse variation", as well as the phrase "y is a function of x". From a strictly logical point of view, we could avoid such usages, but we have preferred to note the way in which they can be interpreted in terms of our definition.

24-5. Summary and Review

Supplementary References

Much interest has been expressed in supplementary reading material in mathematics for the purpose of reference, enrichment, motivation of interest in mathematics, and in materials that may challenge the better students. Access to such materials has not always been easy, and when such reading matter was available, it was incumbent upon the good teacher to direct relevant literature of the appropriate level to the student's attention.

The following list of references for the ninth grade student is offered to help relieve the teacher of this somewhat onerous task. It draws on the volumes of the NEW MATHEMATICAL LIBRARY (NML), which is a series of expository monographs produced by the School Mathematics Study Group and aimed at the level of maturity of the secondary school pupil. The detailed references are by no means complete, but we hope that they will serve to suggest procedures which will be helpful in whetting appetites for mathematical inquiry.

VOLUMES OF THE NEW MATHEMATICAL LIBRARY TO DATE

Volume	Author and Title
1	Niven, Ivan NUMBERS: RATIONAL AND IRRATIONAL
2	Sawyer, W. W. WHAT IS CALCULUS ABOUT?
3	Beckenbach, E. and R. Bellman AN INTRODUCTION TO INEQUALITIES
4	Kazarinoff, N. D. GEOMETRIC INEQUALITIES
5	Salkind, C. T. THE CONTEST PROBLEM BOOK
6	Davis, P. J. THE LORE OF LARGE NUMBERS
7	Zippin, Leo USES OF INFINITY

Volume	Author and Title
8	Yaglom, I. M. GEOMETRIC TRANSFORMATIONS
9	Olds, C. D. CONTINUED FRACTIONS
10	Ore, Oystein GRAPHS AND THEIR USES
11	HUNGARIAN PROBLEM BOOK I
12	HUNGARIAN PROBLEM BOOK II
13	Aaboe, Asger EPISODES FROM THE EARLY HISTORY OF MATHEMATICS
14	Grossman, I. and W. Magnus GROUPS AND THEIR GRAPHS

Following each topic listed below is a set of number pairs such as (1 , 3). The first numeral refers to the volume in the series, and the second, in most cases, to the chapter in that volume. Thus, (1 , 3) refers to Volume 1, Chapter 3. In the case of Volume 6, which is divided by section instead of by chapter, the second numeral refers to the section specified. Volumes 11 and 12 are collections of problems of the Eötvös Competitions for the years 1894 through 1928. These are printed in chronological order. For these, the reference (11 , 1899/3), for example, indicates Volume 11, Problem 3 of the 1899 competition. Those references which we consider to be challenging for the average ninth grader have an asterisk to the left of the reference designation; we leave it to the discretion of the teacher to decide whether the reference is appropriate for the particular student.

Topic	References
Approximation	(6 , 10), (6 , 11), (6 , 12), (12 , 1928/1)
Approximation to $\sqrt{2}$	(7 , 2)
Computing machines	(6 , 15)
Equivalence relation	(10 , 7)
Graphs	
Graphical representation of velocity	(2 , 2)

Properties of numbers	
Associative property	(1 , 1)
Closure	(1 , 1), (1 , 2), (1 , 4)
Commutative property	(1 , 1)
Distributive property	(1 , 1)
Proofs	(1 , 2)
Quadratic equations	*(11 , 1896/2), *(11 , 1899/2)
Topics from physics	
Acceleration	(2 , 7)
Constant speed	(2 , 2)
Instantaneous velocity	(2 , 3)
Scientific notation	(6 , 8)
Varying speed	(2 , 2)
Unique factorization	(1 , Append. B)

SUGGESTED TEST ITEMS

The following suggested test items for each chapter are provided as samples of appropriate questions on the goals of the chapter, rather than as a complete test.

Additional test items can be found in the Teacher's Commentary for the First Course in Algebra. Page references for these will be given for each chapter, using the form:

T.C., 9F, pages 16-18, Items 1-11.

Chapter 1. SETS AND THE NUMBER LINE

True - False:

1. The set of counting numbers is a subset of the set of whole numbers.
2. The set of whole numbers is a subset of the set of rational numbers.
3. The set of counting numbers is a subset of the set of rational numbers.
4. The set of rational numbers is a subset of the set of counting numbers.
5. Let $S = \{0,1\}$. Then S is closed under the operation of forming all possible sums of pairs of elements (including the sum of an element and itself).
6. A line is an infinite set of points.
7. We assume that every point on the number line to the right of the point with coordinate 0 has a coordinate which is a rational number.
8. The set of points on the number line between the point with coordinate $\frac{.998}{1000}$ and the point with coordinate $\frac{999}{1000}$ is an infinite set.
9. There are infinitely many counting numbers between 0 and 1000.

Multiple Choice:

10. Which of the following is not a fraction?

    a. $\frac{10}{2}$ b. [illegible] c. [illegible] d. $\frac{13}{4}$

11. Which of the following [illegible] not name a rational number?

    a. $\frac{[illegible]}{0}$ b. [illegible] c. $\frac{10}{[illegible]}$ d. $\frac{1}{4}$

12. Three of the following numerals name the same rational number. Which one names a different number?

a. $\frac{7}{2}$ b. $\frac{42}{12}$ c. $\frac{56}{14}$ d. $\frac{10.5}{3}$

13. Let W be the set of whole numbers and let N be the set of counting numbers. Of these sets, the number 0 is an element of

a. W only b. N only c. Both W and N d. Neither W nor N

14. Let $R = \{0,1\}$ and let $S = \{2,3\}$. Then $R \cap S$ is

a. $\emptyset$ b. 0 c. $\{0\}$ d. $\{0,1,2,3\}$

15. Which of the following is an infinite set?

a. $\{3,6,9, \ldots, 3000\}$
b. $\{30,60,90, \ldots\}$
c. The set of grains of wheat grown in the United States during 1962.

16. Which of the following is a correct way to write "the set of all even whole numbers"?

a. $\{2,4,6,8, \ldots\}$
b. $\{0,2,4,6, \ldots\}$
c. $\{0,2,4,6,8,\}$
d. $\{2,4,6,8,10\}$

17. Which of the following is a correct way to write "the empty set"?

a. 0 b. $\{0\}$ c. $\emptyset$ d. $\{\emptyset\}$

18. Given the following sets:

D: The set of all counting numbers less than 5.
E: The set of all whole numbers greater than 2.
F: The set $\{1,2,3,4\}$.

Which of the following statements is true?

a. D is a subset of E.
b. E is a subset of D.
c. F is a subset of E.
d. E is a subset of F.
e. D and F are the same set.

19. Let $R = \{1,2,3\}$. Which of the following statements is not true?

a. R is a subset of the set of counting numbers.
b. R is a subset of R.
c. The empty set is a subset of R.
d. R is a subset of the empty set.

20. Two numbers between $\frac{5}{4}$ and $\frac{3}{2}$ are

a. $\frac{11}{8}$ and $\frac{13}{8}$ c. $\frac{21}{16}$ and $\frac{23}{16}$

b. $\frac{17}{12}$ and $\frac{19}{12}$ d. $\frac{23}{20}$ and $\frac{27}{20}$

Completion:

21. The set of people of the United States ________ (is, is not) a subset of the set of people of Washington, D.C.

22. Let $T = \{2,20,28,36\}$ and let $R = \{0,2,4,6, \ldots\}$. Then T ________ (is, is not) a subset of R.

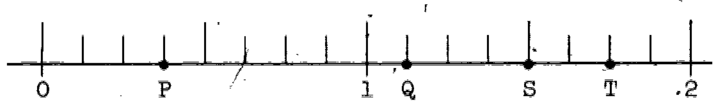
23. Let U be the set of all odd whole numbers which are greater than 100. In set notation U = ______.

24. A number associated with a point on the number line is called the ______ of the point.

25. On the number line the point with coordinate $\frac{13}{3}$ is to the ________ (right, left) of the point with coordinate $\frac{129}{30}$.

Number Line:

DIRECTIONS: Questions 26-28. Refer to the number line and the information given below. Then match the points with the corresponding coordinates.



Points P, Q, S, and T are shown on the number line above. The coordinates of P, Q, S, and T are among the members of the set

$$\{\tfrac{7}{4}, \tfrac{3}{8}, \tfrac{5}{8}, \tfrac{9}{8}, \tfrac{15}{8}, \tfrac{15}{16}\}.$$

26. The coordinate of ______ is $\frac{9}{8}$.

27. The coordinate of S is ______.

28. The coordinate of ______ is $\frac{7}{4}$.

29. On the number line show the graph of the set of counting numbers between $2\frac{1}{2}$ and $5\frac{1}{2}$.

30. On the number line show the graph of the set of whole numbers less than 3.

DIRECTIONS: Questions 31-34. Match the sets in Column A with the corresponding graphs in Column B.

<u>Column A</u>

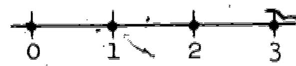
31. The set of counting numbers less than 2.

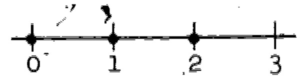
32. The set of numbers in $\{0,1,2\} \cap \{0,1\}$.

33. The set of numbers in $\{0,1,2\} \cup \{0,1\}$.

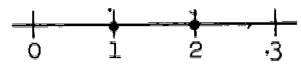
34. The set of whole numbers between 0 and 3.

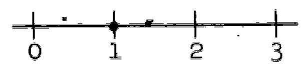
<u>Column B</u>

a. 

b. 

c. 

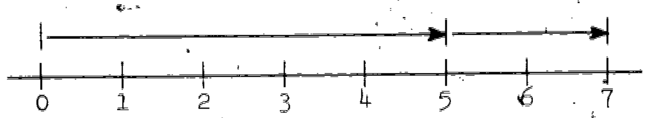
d. 

e. 

35. Which of the following is illustrated by the diagram below?

a. $2 \times (2 + 2)$ b. 2×3 c. 3×2 d. $2 \times 2 \times 2$

36. Which of the following is illustrated by the diagram below?



a. $5 + 7$ b. $5 + 2$ c. $2 + 5$ d. $2 + 7$

Also, see T.C., 9F, pages 16-18, Items 1-11.

Chapter 2. NUMERALS AND VARIABLES

True-False:

1. Addition and multiplication are binary operations.
2. A symbol for an operation is a numeral.
3. A given numeral may represent several numbers.
4. A given number may be represented by several numerals.
5. $\frac{6(7 - 3)}{15 - 3}$ is a name for the number 2.
6. $5(1 + 1) = 5 + 1$
7. $\frac{12 - 5}{5} = 12 - 1$

Matching:

DIRECTIONS: Questions 8-13. Select the property that is illustrated by the example. A property may be selected more than once.

	Example		Property
8.	$\frac{1}{3} + \frac{1}{5} = \frac{1}{5} + \frac{1}{3}$	a.	Associative property of addition
9.	$(4 + 5) + (8 + 2) = (8 + 2) + (4 + 5)$	b.	Associative property of multiplication
10.	$(5 \times 7) \times 2 = 5 \times (7 \times 2)$	c.	Commutative property of addition
11.	$(\frac{2}{5} + \frac{1}{6}) + \frac{5}{6} = \frac{2}{5} + (\frac{1}{6} + \frac{5}{6})$	d.	Commutative property of multiplication
12.	$(5 \times 2) + (3 \times 2) = (5 + 3)(2)$	e.	Distributive property
13.	$5 \times 2.6 = 2.6 \times 5$		

DIRECTIONS: Questions 14-17. Match the items in Column B with the corresponding items in Column A.

	Column A		Column B
14.	$2 + 5 = 7$	a.	Numerical phrase for the number 22
15.	$5 + 4 = 7$	b.	A true sentence
16.	$(30)(5)$	c.	Indicated product
17.	$18 + 4$	d.	A false sentence
		e.	Commutative property of addition

Multiple Choice:

18. **Select the true sentence:**

a. $(\frac{5}{7} + \frac{1}{4})(28) = \frac{5}{7} + \frac{1}{4}(28)$

b. $(\frac{5}{7} + \frac{1}{4})(28) = \frac{5}{7}(28) + \frac{1}{4}$

c. $(\frac{5}{7} + \frac{1}{4})(28) = 20 + 7$

d. $(\frac{5}{7} + \frac{1}{4})(28) = 7\frac{5}{7}$

19. **The expression** $\frac{2(5 + 3)}{4}$ indicates

a. The quotient of the numbers 2×5 and $\frac{3}{4}$.

b. The quotient of the numbers $2(5 + 3)$ and 4.

c. The sum of the numbers 2×5 and $\frac{3}{4}$.

d. The sum of the numbers $\frac{5}{2}$ and 3.

20. **By "the distributive property" we mean "the distributive property**

a. of addition over multiplication". b. of multiplication over addition".

21. Given the sentences:

I. $3(n + 4) = 3n + 4$

II. $3n + 4 = 4 + 3n$

III. $3(n + 4) = (n + 4)3$

IV. $3(n + 4) = 3n + 12$

For n any number of arithmetic, the following are true:

a. I and II only

b. II and III only

c. II and IV only

d. II, III, and IV

22. Given the sentences:

I. $n + 3 = 5 + n$

II. $(n + 3) + 5 = n + (3 + 5)$

III. $5(n + 3) = 5n + 3$

If n is zero, then the following is true

a. I only b. II only c. III only d. Both II and III

Completion:

23. By the distributive property

$$(\frac{3}{4} + \frac{1}{3})12 + (\frac{3}{4} + \frac{1}{3})13 = (\frac{3}{4} + \frac{1}{3})(\_\_\_\_\_)$$

24. By the distributive property

$$(\frac{2}{9} + \frac{3}{5})(45) = \_\_\_\_\_(45) + \_\_\_\_\_(45)$$

25. Use the distributive property to express the numerical phrase $45(\frac{3}{5} + \frac{7}{9})$ as an indicated sum. _____

26. Use the distributive property to express the phrase $\frac{3}{5}(18) + \frac{3}{5}(17)$ as an indicated product. _____

27. Insert parentheses to make a true statement.

$$6 + 13 \times 5 = 95 \quad \_\_\_\_\_$$

Finding Common Names for Numbers:

DIRECTIONS: Questions 28-31. Find the common name of each of the following. Hint: If you use the properties of addition and multiplication, you may be able to write the answer at sight.

28. $(7)(13)(\frac{1}{7}) =$ _____

29. $(\frac{1}{2})(57) + (9\frac{1}{2})(57) =$ _____

30. $6\frac{3}{5} + 20 + 3\frac{2}{5} + 18 =$ _____

31. $12(\frac{2}{3} + \frac{3}{4}) =$ _____

DIRECTIONS: Questions 32-34. Find the common name for each of the following.

32. $\frac{2(a + b)}{3}$ when a is 5 and b is 4.

33. $c(a + b) + c$ when a is 3, b is 2, and c is 5.

34. $(a + b)(c + d)$ when a is 2, b is 1, c is 3, d is 4.

Also, see T.C., 9F, pages 40-42, Items 1-16.

Chapter 3. SENTENCES

True-False:

1. $2(3 + 5) \leq 2(3) + 5$

2. $(7 - 2)3 < 21 - 2$

3. $(1 + 2)(2 + 3) < 3(5)$

4. $5 + 3 \geq 8$

5. $5 = 3 + 2$ or $5 > 4 + 3$

6. $5 = 3 + 2$ and $5 > 4 + 3$

7. $7 > 5 + 1$ and $4 < 2 + 3$

8. $7 < 5 + 1$ or $4 + 1 < 2 + 3$

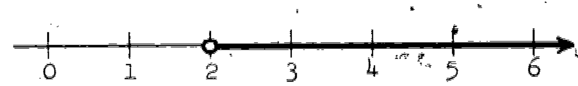
Completion:

9. The truth set of an open sentence is a subset of the _______ of the variable.

10. The graph of a sentence is a graph of the _______ _______ of the sentence.

11. The sentence "$3 < n < 5$" may be read "$n > 3$ _______ (and,or) $n < 5$".

12. Write an open sentence whose truth set is graphed below.



_______.

13. Given the formula $F = \frac{9}{5}C + 32$, if C is 45, then F is _______.

Multiple Choice:

14. Given the sentences

I. $\frac{1}{2}x + 3 = 5$ II. $4t + 6 \leq 22$

Then $\{4\}$ is the truth set of

a. I, but not II
b. II, but not I
c. Both I and II
d. Neither I nor II

15. The open sentence "$p \leq 10$" has the same truth set as

a. $p < 10$ and $p = 10$ c. $p < 10$ or $p = 10$

b. $p > 10$ and $p = 10$ d. $p > 10$ or $p = 10$

16. The open sentence "$t \nless 6$" has the same truth set as

a. $t < 6$ and $t = 6$ c. $t < 6$ or $t = 6$

b. $t > 6$ and $t = 6$ d. $t > 6$ or $t = 6$

Finding Truth Sets:

If the domain of the variable is the set $\{0,1,2,3\}$, find the truth set of each open sentence.

17. $y + 5 < 7$

18. $t \ngtr 2$

19. $3t + 2 = 8$

Find the truth set of each of the following open sentences:

20. $t + 3 = 3 + t$

21. $2x + 3 = 4$

22. $t = t^2$

23. $9 - x^2 = 0$

Matching:

DIRECTIONS: Questions 24-28. Select the correct graph for each open sentence. The domain of the variable is the set of all numbers of arithmetic.

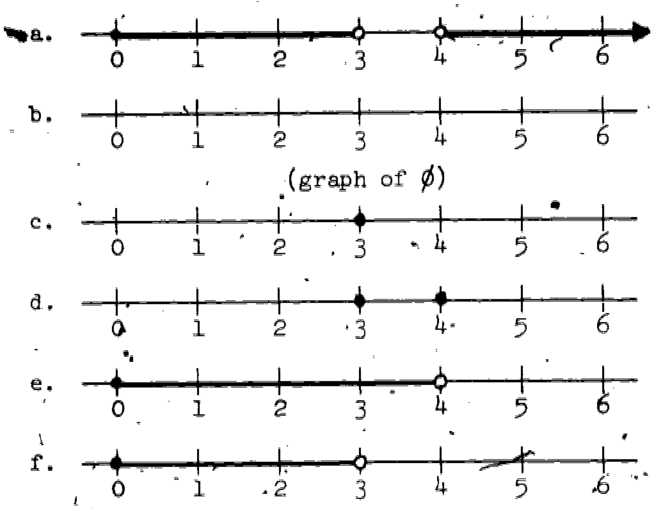
24. $x = 3$ or $x = 4$

25. $t < 3$ and $t < 4$

26. $t < 3$ and $t > 4$

27. $y < 3$ or $y < 4$

28. $y < 3$ or $y > 4$



DIRECTIONS: Questions 29-31. Select the correct truth set for the given sentence. The domain of the variable is the set of all numbers of arithmetic.

Sentences		Truth Set
29. $5x < 25$	a.	$\{0,1,2,3,4\}$
30. $y + 3 > 8$	b.	The set of all numbers of arithmetic except 5
31. $y + 3 \neq 8$	c.	The set of all numbers of arithmetic less than 5
	d.	The set of all numbers of arithmetic greater than 5

Drawing Graphs:

DIRECTIONS: Questions 32-34. Draw the graph of the truth set of each of the following. The domain of the variable is the set of all numbers of arithmetic.

32. $3p - 2 = 4$

33. $2t + 5 = 11$

34. $x \leq 2$

Also, see T.C., 9F, pages 69-72, Items 1-16.

Chapter 4. PROPERTIES OF OPERATIONS

True-False:

1. $787 + (13 - 13) > 787$
2. $787(13 - 13) < 787$
3. $5(0) = 0$ and $5 + 0 = 0$
4. $5(0) = 5$ or $5 + 0 = 5$
5. $\dfrac{1 + \frac{1}{3}}{\frac{1}{7} + 1} = \dfrac{21 + 7}{3 + 21}$
6. $7(x + 3) < 7x + 3$
7. $8xy + 6xy = 14xy$
8. $3t + 4x + 5x + 2t = 5t + 7x$

Completion:

9. Since the sum of any number n and zero is equal to n, zero is called the _______ element for addition.
10. The multiplication property of 1 may be used to write
$$\frac{7}{9} = \frac{7}{9}(\quad) = \frac{56}{72}.$$
11. The common name for $\dfrac{\frac{2}{3} + 15}{15}$ is _______.
12. The distributive property may be used to write
$$(x + 3)(x + 7) = (x + 3)\,\_\_\_\_ + (x + 3)\,\_\_\_\_.$$
13. The sentence $3 \div 5 = 5 \div 3$ is not true. This shows that the _______ property does not hold for division.
14. We are assured that the phrase $(\frac{3}{4})(0)$ names the number 0 because of the _______ property of _______.

Multiple Choice:

15. The truth set of the sentence $t(t - t) = t$ is

a. The set of all numbers of arithmetic
b. The set of all whole numbers
c. $\emptyset$
d. $\{0\}$

16. Let $R = \{0, \frac{1}{2}, 1\}$ and $S = \{\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \frac{5}{2}, \ldots\}$. Which of these is(are) closed under the operation of forming all possible sums of pairs of elements (including the sum of an element and itself)?

a. R only b. S only c. Both R and S d. Neither R nor S

17. Consider the sentences

I. $(3 + x)y = 3y + xy$
II. $x + (5 - 5) = y$

Which of these is(are) true for every value of the variables?

a. I only b. II only c. Both I and II d. Neither I nor II

18. If the properties of addition and multiplication are used, the open phrase $7x + 3y + 5y + x$ can be written as ____

a. $7x + 8y$
b. $7x^2 + 8y$
c. $8x + 8y$
d. $8x^2 + 8y^2$

19. If the properties of operations are used, $4y + 6(x + y)$ can be written as

a. $6x + 5y$
b. $6x + 10y$
c. $6x + 30y$
d. $10x + 10y$

20. If the properties of operations are used, $(2 + x)(5 + x)$ can be written as the indicated sum

a. $10 + x^2$
b. $10 + 2x + x^2$
c. $10 + 5x + x^2$
d. $10 + 7x + x^2$

21. If the distributive property is used, the indicated sum $xy + xy^2$ can be written as the indicated product

a. $(xy + 1)xy^2$
b. $xy(1 + y)$
c. $xy(1 + y^2)$
d. $xy(1 + xy^2)$

Matching:

DIRECTIONS: Questions 22-26. Select the property that is illustrated by the example.

22. $2x + (5y + 5x) = 2x + (5x + 5y)$
23. $2x + (5x + 5y) = (2x + 5x) + 5y$
24. $(2x + 5x) + 5y = (2 + 5)x + 5y$
25. $\frac{1}{2}(bh) = \frac{1}{2}(hb)$
26. $\frac{1}{2}(hb) = (\frac{1}{2} \times h)b$

a. Associative property of addition
b. Associative property of multiplication
c. Commutative property of addition
d. Commutative property of multiplication
e. Multiplication property of 1
f. Distributive property

Use of Properties:

DIRECTIONS: Questions 27-28. Using the distributive property write each indicated product as an indicated sum without parentheses.

27. $(2 + t)t$
28. $3a(5b + 4c)$

DIRECTIONS: Questions 29-31. Using the distributive property write each indicated sum as an indicated product.

29. $3x + 3y$
30. $ay + by$
31. $7y + 2y^2$

DIRECTIONS: Questions 32 and 33. Using the properties of addition and multiplication write each open phrase in simpler form.

32. $23x + 2y + 10x$
33. $3a + b + 2 + 4b + 8a$

Also, see T.C., 9F, pages 72-74, Items 17-26.

Chapter 5. OPEN SENTENCES AND ENGLISH SENTENCES

Writing Open Phrases for English Phrases:

DIRECTIONS: Questions 1-13. Write an open phrase suggested by each of the following English phrases. Use only the given variables.

1. The number of quarters in t dollars.
2. The cost in cents of k gallons of gasoline at 35.9 cents per gallon.
3. The cost in cents of n candy bars at 6 bars for 25 cents.
4. Three-fourths of the number of questions on an algebra test consisting of y questions.
5. The successor of the whole number n.
6. Five more than $\frac{2}{3}$ of the number n.
7. The total number of cents in k quarters, d dimes, and n nickels.
8. The difference of n and 5 if $5 > n$.
9. The number x decreased by three times another number y.
10. The product of 5 and the sum of 3 and y.
11. The number of square feet in the area of a square if s represents the number of feet in the side of the square.
12. The number of inches in the perimeter of a rectangle, if t is the number of inches in the length and the length is twice the width.
13. The smaller of two numbers if the sum of the numbers is 35 and the larger number is z.

Matching Open Phrases and English Phrases:

DIRECTIONS: Questions 14-21. In each of the following select an English phrase which might be represented by the given open phrase.

14. $3n$

15. $\frac{n}{3}$

16. $12n$

17. $\frac{n}{12}$

18. $2n + 1$

19. $10n + 5(4n)$

20. $n - 15$

21. $15 - n$

a. Cost in cents of one orange at n cents per dozen

b. Annual salary [illegible] at n dollars per month

c. The successor of [illegible] number $2n$

d. Two more than the number n

e. The number of feet in n yards

f. The number of yards in n feet

g. The number of cents in n dimes and $4n$ nickels.

h. John's age in years, n years ago, if he is now 15 years old.

i. The number that is 15 less than n.

Writing Open Sentences for English Sentences:

DIRECTIONS: Questions 22-32. Write an open sentence for each English sentence.

22. The number of nickels n in John's pocket is not less than 5.

23. Five less than the number n is 32.

24. The number of inches in the side of a square is s and its perimeter is 35 inches.

25. In a class of 27 students the number of boys, b, is 3 more than the number of girls.

26. The sum of two-thirds of a number n and three-fourths of the same number is 51.

27. The sum of the whole number w and its successor is not equal to 100.

28. John has more than 15 coins in his pocket of which there are n nickels and 3 times as many dimes as nickels.

29. John has less than 89 cents consisting of n nickels and 5 more dimes than nickels.

30. The number of questions n on the English test is less than 50 and greater than 40.

31. The length of a rectangle is 3 inches more than twice the width and the area is 35 square inches.

32. Florie bought 30 stamps, some of them 5-cent stamps and some 8-cent stamps, and spent $1.71.

See also T.C., 9F, pages 99-107, Items 1-15.

Chapter 6. THE REAL NUMBERS

True-False:

1. The opposite of the opposite of a negative number is a negative number.

2. For every real number r, the opposite of r is less than or equal to r.

3. For every real number r, the absolute value of r is greater than the opposite of r.

4. For every positive real number r, $|-r|$ is positive.

5. The set of all negative rational numbers is a subset of the set of all numbers of arithmetic.

6. The set of all rational numbers is a subset of the set of all real numbers.

7. $-\sqrt{2}$ is a real number.

8. $-\sqrt{2}$ is a negative rational number.

9. If a is a real number and b is a real number, then exactly one of the following is true:

$$a \leq b, \quad b < a.$$

10. If a is a real number and b is a real number, then exactly one of the following is true:

$$a \neq b, \quad b \leq a, \quad a \leq b.$$

Completion:

11. Consider $S = \{b, -b\}$ where $b \neq 0$. The greater of the two elements in S is called the ______ ______ of b.

12. Each negative number is the ______ of a positive number.

13. For every positive real number b, $b + |b| =$ ______.

14. The set of all non-zero real numbers ______ (is, is not) closed under the operation of taking the opposite of each element.

15. For all real numbers a, b, and c, if a is the opposite of b, and b is the opposite of c, then a is ______ c.

16. We know that $2 < 3$. Make an order statement about the opposites of 3 and 2. ______

17. Write the common numeral for the opposite of $|-9|$. ______ .

18. Write an order statement connecting -2 and the opposite of -2. ______

19. Consider the two points with coordinates $-\pi$ and -3.5. The point with coordinate $-\pi$ is to the ______ (right, left) of the point with coordinate -3.5.

20. Write a sentence having the same meaning as "$a \not> -3$" using the symbol "$\geq$". ______

Multiple Choice:

21. Which of the following numbers is the greatest?

a. The opposite of 0.28.
b. The opposite of 0.25.
c. The opposite of -0.28.
d. The opposite of -0.25.

22. If t is negative, which of the following is negative?

a. $-t$ b. $|t|$ c. $|-t|$ d. $-(-t)$

23. If we know that v, x, and y are real numbers with $|v| = |x|$ and $|x| = |y|$, then we may be sure that

a. $v = y$
b. $v = y$ or $v < y$
c. $v = y$ or $v > y$
d. $v = y$ or $v = -y$

24. Which of the following has the same truth set as the open sentence $(-x) = 4$?

a. $|x| = 4$ b. $|-x| = 4$ c. $x = (-4)$ d. $x \neq 4$

25. Which one of the following has the same truth set as the open sentence $x \not< 0$?

a. $x > 0$ b. $x \geq 0$ c. $x \neq 0$ d. $x = 0$

26. Given the open sentences:

I. $(-3) < x$ and $5 < x$
II. $(-3) < x$ and $5 > x$
III. $(-3) > x$ and $5 > x$
IV. $(-3) > x$ and $5 < x$

Then $\emptyset$ is the truth set of

a. I and II only b. III and IV only c. II only d. IV only

Truth Sets of Open Sentences:

27. Find the truth set of the open sentence $-(-y) = 5$.

28. Find the truth set of the open sentence $|-x| = 5$.

Graphs:

29. Draw the graph of the set of integers less than 4 and greater than -4.

DIRECTIONS: Questions 30-33. Draw the graph of the truth set of each of the following open sentences.

30. $-y \leq 0$

31. $-x > -2$

32. $-(-x) < 2$

33. $-|x| < 0$

Also, see T.C., 9F, pages 137-138, Items 1-14.

Chapter 7. PROPERTIES OF ADDITION

Finding Common Name:

DIRECTIONS: Questions 1-10. Find the common name for each of the following.

1. $(-20) + 6$
2. $(-5) + (-3)$
3. $-(-3) + (-3)$
4. $(-4) + \big(15 + (-15)\big)$
5. $-|5| + |-5|$
6. $(-8) + 5 + (-8)$
7. $|(-8) + 5| + (-3)$
8. $-\big((-3) + 2\big)$
9. $-\big((-3) + (-2)\big)$
10. $-\big(|-3| + |-2|\big)$

Multiple Choice:

11. Each real negative number has how many additive inverses?

    a. none b. one c. two d. It depends on the number.

12. Each real non-negative number has how many additive inverses?

    a. none b. one c. two d. It depends on the number.

13. If $(-\frac{15}{4}) + n = (-\frac{27}{2})$ is true for some number n, then n is

    a. $-\frac{39}{4}$ b. $-\frac{69}{4}$ c. $\frac{39}{4}$ d. $\frac{69}{4}$

14. If $|x| > x$, then x is

    a. negative c. non-negative
    b. positive d. any real number except zero

15. Which of the following names a negative number?

    a. $|(-7) + (-9)|$ c. $(-7) + |-9|$
    b. $|-7| + |-9|$ d. $|-7| + (-9)$

16. Let $R = \{-1, 0, 1\}$ and $S = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$. Which set is closed under the operation of forming all possible sums of pairs of elements (including the sum of an element and itself)?

a. R only b. S only c. Both R and S d. Neither R nor S

17. Consider the statements:

I. $(-[illegible]) + 0 = (-[illegible])$

II. $(-[illegible]) + (-[illegible]) = -([illegible] + [illegible])$

III. $-\big([illegible] + (-10)\big) + \big([illegible] + (-10)\big) = 0$

Which of these illustrate(s) the addition property of opposites?

a. I only b. II only c. III only d. II and III

18. The following statements are true for every x, every y, and every z.

I. $x + (-x) = 0$

II. If $x = [illegible]$, then $x + z = y + z$

III. $-(x + y) = (-x) + (-y)$

Which of the statements express(es) the addition property of equality?

a. I only b. II only c. III only d. I and II

19. Let [illegible] be a positive real number and [illegible] a negative real number. The sum $x + y$ is negative if

a. $|x| + |y| > 0$ c. $|y| - |x| > 0$

b. $|x| - |y| > 0$ d. $|y| - |x| [illegible] 0$

20. If [illegible] and [illegible] are negative real numbers, which of the following is not true?

a. $(-[illegible]) + (-[illegible]) [illegible] |[illegible] + [illegible]|$ c. $(-[illegible]) + (-[illegible]) < |[illegible]| + [illegible]$

b. $(-[illegible]) + (-[illegible]) [illegible] |[illegible]| + |[illegible]|$ d. [illegible] $+ (-[illegible])$

Finding Truth Sets:

DIRECTIONS: Questions 21-25. Find the truth sets of each of the following open sentences.

21. $12 + t = (-4)$

22. $(-12) + t = (-4)$

23. $(-12) + (-t) = (-4)$

24. $|x| + 5 = 8$

25. $23 + |-x| = 10$

Completion:

26. What is the distance between -7 and 3 on the number line? _______

27. What is the distance between -8 and 8 on the number line? _______

28. If the sentence $5 + (-\frac{13}{2}) = y + \frac{3}{4}$ is true for some number y, then $5 + (-\frac{13}{2}) +$ _______ $= y$ is true for the same y.

29. $\left((-7) + 11\right) + (-11) = (-7) + \left(11 + (-11)\right)$ is true because of the _______ property of _______.

30. $(7 - 13) + \left(6 + (-17)\right) = (-13) + \left((-17) + 6\right)$ is true because of the _______ property of _______.

31. If $x + (-3) = 7$ is true for some number x, then $\left(x + (-3)\right) + 3 = 7 + 3$ is true for the same number x because of the _______ property of _______.

See, also, T.C., 9F, pages 167-169, Items 1-9.

Chapter 8. PROPERTIES OF MULTIPLICATION

Finding Common Name:

DIRECTIONS: Questions 1-8. Write the common name for each of the following.

1. $(-4)(23)(-\frac{1}{4})$

2. $(-7)^2$

3. $-(7^2)$

4. $(-5)(2)(-3)$

5. $|(-5)(2)|(-3)$

6. $(-8)\left(7 + (-7)\right)$

7. $(\frac{5}{3})(-7) + (-\frac{2}{3})(-7)$

8. $-(-\frac{2}{3})(-8)(-\frac{3}{2})$

Completion:

9. Simplify the expression $(-3a^2b)(4ab)$. ______

10. If the product of the real numbers t and r is zero and t is negative, then r is ______.

11. $(-5)\left((-6)(-\frac{1}{5})\right) = (-5)\left((-\frac{1}{5})(-6)\right)$ is true because of the ______ property of ______.

12. $(-5)\left((-\frac{1}{5})(-6)\right) = \left((-5)(-\frac{1}{5})\right)(-6)$ is true because of the ______ property of ______.

13. If $3x = -39$ is true for some number x, then $(3x)(\frac{1}{3}) =$ ______ is true for the same number x because of the multiplication property of equality.

14. $(-5x) + 2x^2 + 3x =$ ______.

15. $3x + (-6y) + (-4x) + (-y) =$ ______.

DIRECTIONS: Questions 16-18. Write each of the following indicated products as an indicated sum, collect terms and simplify.

16. $\left(4x + (-3y) + 5\right)(-3y) =$ ______.

17. $\left((-x) + 1\right)(x + 1) =$ ______.

18. $(5x + 2)\left(x + (-1)\right) =$ ______.

Multiple Choice:

19. Which of the following is a correct application of the distributive property? $(-6)(-12) + (-6)(5) =$

a. $(-6)(-12) + 5$

b. $(-12)(-6) + (5)(-6)$

c. $(-6)(5) + (-6)(-12)$

d. $(-6)\left((-12) + 5\right)$

20. Which of the following is a correct way to write the opposite of $\left(x + (-y)\right)$ as an indicated sum?

a. $(-y) + x$

b. $x + y$

c. $(-x) + y$

d. $(-x) + (-y)$

See, also, T.C., 9F, pages 210-212, Items 1-4, 6, 8.

Chapter 9. MULTIPLICATIVE INVERSE

Finding Common Name:

1. $\frac{1}{(-2)} \cdot \frac{1}{(-4)} =$ _______.

2. $\frac{1}{\frac{1}{(-2)}} =$ _______.

Multiple Choice:

3. If $n < -1$ and t is the multiplicative inverse of n, which of the following sentences is true?

   a. $t + n = 1$ b. $t + n \geq 0$ c. $n > t$ d. $t > n$

4. The number x is a solution of $5 + 4x = x + 6$ if and only if x is a solution of

   a. $3x + 1 = 0$ b. $3x = 1$ c. $x + 3 = 0$ d. $x = 3$

5. The sentence $2x + 5 = 1$ is equivalent to

   a. $x + 2 = 0$ b. $x + 3 = 0$ c. $x = 2$ d. $x = 3$

6. If p is positive and n is negative, which of the following sentences is not true?

   a. $\frac{1}{p} > 0$ b. $\frac{1}{p} > \frac{1}{n}$ c. $\frac{1}{p} \cdot \frac{1}{n} > 0$ d. $0 > \frac{1}{n}$

7. If n is negative and t is the reciprocal of n, which of the following sentences is not true?

   a. $nt = 1$ b. $n + t = 0$ c. $\frac{1}{t} = n$ d. $\frac{1}{n} \cdot \frac{1}{t} = 1$

DIRECTIONS: Questions 8-12. Find the truth set of each of the following.

8. $2 + 3x = 0$

9. $8x + (-3) = (-11)$

10. $(-\frac{2}{3})x = 2$

11. $x(x + 1) = 0$

12. $\left(y + (-2)\right)(y + 5) = 0$

See, also, T.C., 9F, pages 211-212, Items 5,7, 9-10.

Chapter 10. PROPERTIES OF ORDER

Completion:

DIRECTIONS: Questions 1-4. Select the correct symbol.

1. $(-28) + 213 \underset{(<,=,>)}{\underline{\qquad}} (-30) + 213$

2. $(-1.99)(.99) \underset{(<,=,>)}{\underline{\qquad}} (-2.01)(.99)$

3. $(-1.99)(-.99) \underset{(<,=,>)}{\underline{\qquad}} (2.01)(-.99)$

4. $\left(3 + (-5)\right)^2 \underset{(<,=,>)}{\underline{\qquad}} \left((-5) + 3\right)^2$

Order Statements:

5. If $x < y$ for every choice of the variables, make an order statement connecting $-x$ and $-y$.

6. If $x \leq y$ for every choice of the variables, make an order statement connecting $x + (-3)$ and $y + (-3)$.

7. If $t > 0$ and $y = x + t$, make an order statement connecting x and y.

8. If $0 < y < 1$, make an order statement connecting y and y^2.

9. If $y < 0$, make an order statement connecting y and y^2.

10. If $0 < x < y$, make an order statement connecting x^2 and y^2.

Multiple Choice:

11. Let x be a real number such that $x^2 \not> 0$. Then which of the following is true?

a. $x > 0$ b. $x = 0$ c. $x < 0$ d. $x \neq 0$

12. If $(-\frac{3}{5})x + 5 < 2$ is true for some number x, then which of the following is true for the same x?

a. $(-\frac{3}{5})x + 3 < 0$ c. $(-\frac{3}{5})x > 3$

b. $(-\frac{3}{5})x + 3 > 0$ d. $(-\frac{3}{5})x > -3$

13. Which of the following open sentences has the same truth set as $(-\frac{2}{3})x > 6$?

a. $x \geq -9$ b. $x > -4$ c. $x < -9$ d. $x < -4$

14. The sentence $(-x) + 3 > 2$ is equivalent to

a. $x > -1$ b. $x > 1$ c. $x < -1$ d. $x < 1$

15. For any real numbers r, s, and t, if $r \not< s$ and $s \not< t$, then

a. $r > t$ b. $r \geq t$ c. $r < t$ d. $r \leq t$

16. Which of the following is true if $\frac{1}{x} < \frac{1}{y}$, $x < 0$ and $y < 0$?

a. $x < y$ b. $x > y$ c. $x \leq y$ d. $x = y$

<u>Finding</u> <u>Truth</u> <u>Sets</u>:

17. Find the truth <u>set</u> of $-x \geq 0$.

18. Find the truth <u>set</u> of $(-2) + y \leq 4 + y$.

<u>Graphing</u> <u>Truth</u> <u>Sets</u>:

19. Graph the truth set of $x + 4 < 0$.

20. Graph the truth set of $(-\frac{1}{2})x < 2$.

21. Graph the truth set of $-5x + 3 > 8$.

<u>Writing</u> <u>Open</u> <u>Sentences</u>:

DIRECTIONS: Questions 22 and 23. Write an open sentence for each English sentence. DO NOT SOLVE.

22. The product of two consecutive whole numbers is greater than 100 and the larger number is y.

23. The sum of a number t and its reciprocal is not less than the opposite of the number.

See, also, T.C., 9F, pages 210-212, Items 1-8.

Chapter 11. SUBTRACTION AND DIVISION

Completion:

DIRECTIONS: Questions 1-4. Write the common name for each of the following.

1. $(-28) - 13 =$ ______

2. $15 - (10 - 20) =$ ______

3. $\frac{3}{-15} =$ ______

4. $\frac{-\frac{3}{4}}{\frac{1}{4}} =$ ______

5. What number is 5 less than -8 ? ______

6. What is the distance between -10 and -50 ? ______

7. What is the distance between -10 and 50 ? ______

8. For any real numbers x and y, the difference $x - y$ is the same as the sum of x and ______.

9. For any real numbers x and y, $y \neq 0$, the quotient, x divided by y, is the same as the product of x and ______.

Multiple Choice:

10. Consider the operation of subtraction. Which of the following statements is true?

    a. Subtraction is associative but not commutative.
    b. Subtraction is commutative but not associative.
    c. Subtraction is associative and commutative.
    d. Subtraction is neither associative nor commutative.

11. Consider the operation of division. Which of the following statements is true?

    a. Division is associative but not commutative.
    b. Division is commutative but not associative.
    c. Division is associative and commutative.
    d. Division is neither associative nor commutative.

12. Let R be the set of all real numbers and let S be the set of all negative real numbers. Which of these is(are) closed under the operation of subtraction?

a. R only b. S only c. Both R and S d. Neither R nor S

13. Let R be the set of all real numbers except zero and let S be the set of all negative real numbers. Which of these is(are) closed under the operation of division?

a. R only b. S only c. Both R and S d. Neither R nor S

14. If x is any number for which $|x + 2| < 3$ is true, then which of the following is true?

a. The distance between 2 and x is less than 3.
b. The distance between -2 and x is less than 3.
c. The distance between -2 and -x is less than 3.
d. The distance between 2 and x is less than $|3|$.

15. Which of the following is a correct translation of the sentence, "the distance between x and (-4) is greater than 5"?

a. $|x - 4| > 5$ c. $x - 4 > 5$
b. $|x - (-4)| > 5$ d. $x - (-4) > 5$

Order Statements:

16. If $y - x$ is a negative number, write an order statement connecting x and y.

17. If s, y, and z are real numbers and $y - z$ is a positive number, write an order statement connecting $x - y$ and $x - z$.

See, also, T.C., 9F, pages 267-269, Items 1, a-e, j-k, o, q; 2; 4, a, b, d; 5-, 10-11.

Chapter 12. FACTORS AND DIVISIBILITY

Completion:

1. List the four smallest prime numbers. _______ _______ _______ _______

2. The Fundamental Theorem of Arithmetic tells us that every positive integer except one and the primes has _______ (exactly, at least, more than) one prime factorization.

3. If n is odd, then $(n + 2)^2$ is _______ (even, odd).

4. Find the prime factorization of 120. _______

5. Find two numbers whose product is 120 and whose sum is 29.

_______ _______

6. Find the smallest positive integer y for which 6 is a factor of $12 + 13y$. _______.

Multiple Choice:

7. Which of the following numbers does not have any proper prime factors?

a. 115 b. 143 c. 133 d. 117 e. 127

8. Which of the following is not divisible by 3 ?

a. $3^4(6^2 + 2^3)$ c. $3^4 \cdot 6^2 \cdot 2^3$

b. $(3^4 + 6^2)(2^3)$ d. $3^4 + 6^2 + 2^3$

9. 2^3 is a factor of which of the following?

a. $2^3 + 2^2$ b. $2^3 + 3^2$ c. $2^3 + 4^2$ d. $2^3 + 5^2$

10. 5^4 names the same number as

a. $5 + 5 + 5 + 5$ c. $5 \cdot 5 \cdot 5 \cdot 5$

b. $4 + 4 + 4 + 4 + 4$ d. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$

11. A simple name for $(3^4)(3^2)^3$

a. 3^9 b. 3^{10} c. 3^{12} d. 3^{18} e. 3^{24}

See, also, T.C., 9F, pages 309-310, Items 1-8, 11.

Chapter 13. FRACTIONS

Completion:

1. Find the least common denominator of $\frac{2}{15} - \frac{1}{18} + \frac{5}{4}$. _______

2. Find the product of $-\frac{3}{7}$ and the reciprocal of 3. _______

3. The fraction $\frac{9}{5}$ may be interpreted as the ratio of _______ to _______.

4. When we write the phrase $\frac{y + 5}{y(y + 5)}$ we assume that the domain of y excludes _______ and _______.

Write the common name for:

5. $(\frac{-3}{-8})(\frac{5}{-9}) =$ _______

6. $\dfrac{\frac{2}{3} - \frac{5}{2}}{2} =$ _______

Questions 7-12. Simplify.

7. $\frac{35x}{-7x^2} =$ _______

8. $\frac{1 - x}{x - 1} =$ _______

9. $\frac{5x}{6} + \frac{x}{2} =$ _______

10. $\frac{x^2 + x}{x} =$ _______

11. $\frac{3}{x + 3} \cdot \frac{x + 6}{6} =$ _______

12. $\dfrac{1 + \frac{1}{x}}{1 - \frac{1}{x}} =$ _______

Writing Open Sentences:

13. Write an open sentence for the following problem. DO NOT SOLVE.

If the numerator of the fraction $\frac{5}{7}$ is decreased by x, the value of the resulting fraction is -1. By what amount was the numerator decreased?

Multiple Choice:

14. If x and y are real numbers such that $x < y < 0$, which of the following is positive?

a. $x - y$ b. $\frac{1}{x - y}$ c. $\frac{1}{x}$ d. $\frac{1}{x} - 1$ e. $\frac{\frac{1}{x} - 1}{\frac{1}{y} - 1}$

See, also, T.C., 9F, pages 267-269. Items: 1, f-i, l-n, p, r-w; 3; 4, c, f; 8.

Chapter 14. EXPONENTS

Multiple Choice:

1. Which of the following names the same number as $(5x^{-2} + 4^{-1})^0$ if x is 2 ?

   a. 1 b. $\frac{1}{100} + \frac{1}{4}$ c. $\frac{6}{4}$ d. 0

2. Which of the following names the same number as 2^{-4} ?

   a. -2^4 b. $(-2)^4$ c. $-\frac{1}{2^4}$ d. $(\frac{1}{2})^4$ e. $(-4)(2)$

3. Which of the following names the same number as $3^4 + 3^5$?

   a. 3^9 b. $4(3^4)$ c. 6^9 d. 3^{20}

4. Which of the following names the same number as $\frac{2^3}{3^5}$?

   a. $(\frac{2}{3})^3(\frac{1}{3})^2$ b. $(\frac{2}{3})(\frac{1}{3})^2$ c. $(\frac{2}{3})^{-2}$ d. $(\frac{2}{3})^2$

5. If $2^{10} + x = 5815$ is true for some positive integer x, which of the following is true?

   a. 2 is a factor of x.
   b. 5 is a factor of x.
   c. Both 2 and 5 are factors of x.
   d. Neither 2 nor 5 are factors of x.

Completion:

6. If x and y are numbers for which $2^3 \cdot 2^{-3} = 2^x$ and $2^{-3} \cdot 2^{-3} = 2^y$ are true sentences, then $x + y =$ _______.

7. If u and v are numbers for which $\frac{2^3}{2^{-3}} = 2^u$ and $\frac{2^{-3}}{2^3} = 2^v$ are true, then $\frac{v}{u} =$ _______.

DIRECTIONS: Questions 8-13. Simplify and write answer with positive exponents only. Assume no variable has the value 0.

8. $x^{-2}(x^3 + x^2) =$ _______

9. $(x^{-1} + x)^2 =$ _______

10. $\frac{1}{x^2} - \frac{y}{x} + \frac{x}{y} =$ ________

11. $\frac{x^{-2}y^3}{x^3y^{-3}} =$ ________

12. $\frac{-3x^3y^2}{(-3x^3y)^2} =$ ________

13. $\frac{\frac{5a^3b^2}{3ab^3}}{\frac{2(ab)^2}{3b^2}} =$ ________

See, also, T.C., 9F, page 310, Items 9, 10, e and f.

Chapter 15. RADICALS

Completion:

DIRECTIONS: Questions 1-4. Simplify each of the following.

1. $\sqrt{\frac{5}{3}}\sqrt{\frac{1}{2}}\sqrt{\frac{3}{5}} =$ ______

2. $\sqrt[3]{16x^3} =$ ______

3. $(\sqrt{5} + \sqrt{3})^2 =$ ______

4. $\sqrt{\frac{1}{2}} + \frac{1}{2}\sqrt{50} =$ ______

DIRECTIONS: Questions 5 and 6. Rationalize the denominator.

5. $\frac{\sqrt{5} \times \sqrt{2}}{\sqrt{2}} =$ ______

6. $\sqrt{\frac{1}{3x^2}} =$ ______

DIRECTIONS: Questions 7-9. Simplify. Assume that x and y are positive numbers.

7. $\sqrt{xy^3}\,\sqrt{x^3y^2} =$ ______

8. $(\sqrt{x} + 1)(\sqrt{x} - 1) =$ ______

9. $x\sqrt{4x} + 3\sqrt{x^3} =$ ______

10. The number $\sqrt{2} + 1$ ______ (is, is not) the coordinate of a point on the number line.

11. If $\frac{a}{b}$ is a square root of the positive real number t, then the opposite of $\frac{a}{b}$ ______ (is, is not) a square root of t.

12. For every real number r, $\sqrt{r^2} =$ ______.

13. For every real number r, $\sqrt[3]{r^3} =$ ______.

14. The number .00084 may be written as 8.4×10^{2n} where n is ______.

Multiple Choice:

15. If r is a real number, which of the following names a positive real number?

a. $\sqrt[3]{-r^3}$ b. $\sqrt{(-r)^2}$ c. $-\sqrt{r^2}$ d. $-\sqrt{-r^2}$

16. If 300 is used as a first approximation to $\sqrt{83500}$, the best second approximation is

a. 278 b. 578 c. 289 d. 285

17. If $\sqrt{52} \approx 7.211$ and $\sqrt{5.2} \approx 2.280$, which of the following approximations is incorrect?

(Remember that the symbol "$\approx$" is read "is approximately equal to".)

a. $\sqrt{.0052} \approx .07211$

b. $\sqrt{52000} \approx 228.0$

c. $\sqrt{520} \approx 22.8$

d. $\sqrt{.520} \approx .228$

e. $\sqrt{5200} \approx 72.11$

Matching:

DIRECTIONS: Questions 18-20. DO NOT SIMPLIFY. Match each expression in Column A with the corresponding domain of the variable as described in Column B.

Column A	Column B
18. $\sqrt{\frac{x^3}{x}}$	a. Set of all real numbers.
19. $\sqrt{\frac{x^2}{x}}$	b. Set of all real numbers except zero.
20. $\frac{\sqrt{x}}{\sqrt{x}+1}$	c. Set of all non-negative real numbers.
	d. Set of all positive real numbers.

See, also, T.C., 9F, pages 339-340, Items 1-9.

Chapter 16. POLYNOMIALS AND FACTORING

1. Which of the following is a polynomial of degree 3?

   a. $(x^3 + x)^3$ c. $x^2 + x + 2^3$ e. $3x^2 + 3$

   b. $x^3 + x - (x^2 + x^3)$ d. $2 + x^3$

DIRECTIONS: Questions 2-8. Factor completely.

2. $5x^3 - 20x$

3. $9x^2 - 6xy + y^2$

4. $3m^3 - 6mn + 3m$

5. $3a^2 - 6a + 3$

6. $3m^3 - 6mn + 2m^2n - 4n^2$

7. $3y^2 - 27$

8. Factor $x^2 - 10x + 21$ by completing the square.

DIRECTIONS: Questions 9 and 10. Find the truth set.

9. $2x^2 + 9x = 0$

10. $x^2 - 2x - 15 = 0$

DIRECTIONS: Questions 11-14. Write each expression in common polynomial form.

11. $(2a + 3a^2 - 5) + (7a - a^2 + 11)$

12. $(3b^2 - 2b + 5) - (b^3 + 7b^2)$

13. $3y^2(2 - 4y)$

14. $(a - 5)(a - 1)$

15. A form with rational denominator for $\frac{2}{5 - \sqrt{6}}$ is ______.

T.C., 9F - No reference

Chapter 17. QUADRATIC POLYNOMIALS

Multiple Choice:

1. Consider the polynomials:

   I. $x^2 - 6x - 1$ II. $x^2 - x - 6$ III. $x^2 - x + 6$

   Which of these is factorable over the integers?

   a. I only b. II only c. III only d. I, II, III
   e. none of these

2. Consider the polynomials:

   I. $x^2 - 6$ II. $x^2 + 6$ III. $x^2 + 6x$

   Which of these is factorable over the real numbers but is not factorable over the integers?

   a. I only b. II only c. III only d. I, II, III
   e. none of these

3. Consider the forms:

   I. $x(x + 2) - 1$ II. $(x + 1)^2 - 2$ III. $(x + 1 + \sqrt{2})(x + 1 - \sqrt{2})$

   Which of these is a factored form of $x^2 + 2x - 1$?

   a. I only b. II only c. III only d. I, II, III
   e. none of these

4. Determine the standard form of the quadratic polynomial $x^2 - 5x + 8$.

   a. $\left(x - \frac{5}{2}\right)^2 + \frac{7}{4}$ d. $\left(x - \frac{5}{2}\right)^2 + \frac{21}{2}$

   b. $\left(x - \frac{5}{2}\right)^2 + \frac{57}{4}$ e. $\left(x - \frac{5}{2}\right)^2 + 8$

   c. $\left(x - \frac{5}{2}\right)^2 + \frac{11}{2}$

5. Find the truth set of the equation $(x + 3)^2 - 1 = 0$.

a. {2,4} b. {2,-4} c. {-2,4} d. {-2,-4}

6. Which one of the following quadratic polynomials can be factored over the reals but cannot be factored over the integers?

a. $2x + x^2$

b. $-1 + 2x + x^2$

c. $-1 - 2x - x^2$

d. $2 + 2x + x^2$

e. $2 + x^2$

Factor Completely:

7. $x^2 - 11x + 7$

8. $2x^2 - 3x - 2$

Find the Truth Set of:

9. $5x^2 + 4x = 1$

10. $x^2 + 7 = 6x$

See, also, T.C., 9F, pages 392-393, Items 2, 6, 12
pages 570-571, Items 3, 5, 6

Completion:

1. The domain of x for the expression $\frac{(x + 1)^2}{x^2 - 1}$ does not include the number(s) ______.

2. The expression $\frac{(x + 5)(x - 2)}{x - 2}$ and $x + 5$ name the same number for ______ value(s) of the variable.
(all, some, no)

3. Find the quotient and the remainder if $6x^2 - 5x$ is divided by $2x + 5$.
Quotient is ______, remainder is ______.

DIRECTIONS: Questions 4-6. Simplify. Write your result as a single indicated quotient of two polynomials which do not have common factors.

4. $\frac{(x + 1)^2}{x^2 - 1} =$ ______

5. $\frac{3}{3x - x^2} - \frac{1}{x} =$ ______

6. $\frac{1 - \frac{2}{x + 2}}{x} =$ ______

Multiple Choice:

7. Consider the expressions:

I. $x + |x|$ II. $x + \sqrt{x^2}$ III. $x + x^2$

Which of these is a rational expression?

a. I only b. II only c. III only d. I, II, III
e. none of these

Find the truth set:

8. $\frac{2x}{x + 3} - \frac{15}{x^2 - 9} = 2$

See, also, T.C., 9F, pages 392-393, Items 1, 3-5, 9, 10, 13

Chapter 19. TRUTH SETS OF OPEN SENTENCES

Multiple Choice:

1. An operation which always yields an equivalent sentence is

   a. multiplying both members of the sentence by the same expression.

   b. adding the same real number to both members.

   c. squaring both members.

2. For any real numbers x and y, the sentence $x^2 = y^2$ is equivalent to

   a. $x = y$
   b. $x = -y$
   c. $x = |y|$
   d. $|x| = y$
   e. $(y - x)(y + x) = 0$

3. Which of the following sentences is equivalent to the open sentence $x + 5 = 0$?

   I. $x(x + 5) = 0$ II. $x^2(x + 5) = 0$

   a. I only b. II only c. both I and II d. neither I nor II

4. Which of the following sentences is equivalent to the open sentence $2x - 1 = 0$?

   I. $(2x - 1)^2 = 0$ II. $(2x)^2 = 1$

   a. I only b. II only c. both I and II d. neither I nor II

Find the truth set:

5. $\frac{3}{x - 9} = \frac{2}{x + 2}$

6. $\sqrt{x - 2} + 2 = 5$

7. $(x^2 - 9)(x + 1) = 0$

8. $(x^2 - 9)(x + 1) = 0$ and $x + 1 < 0$

9. $\frac{3}{x^2 - 1} = 1$

10. $2x = \sqrt{x^2} - 1$

11. $x(x - 2) + 2(x - 2) = 0$

12. $x(x - 2) = 2(x - 2)$

DIRECTIONS: Questions 13-15. Graph the truth set of each of the following sentences.

13. $(x^2 + 1)(2 - x) \leq 0$

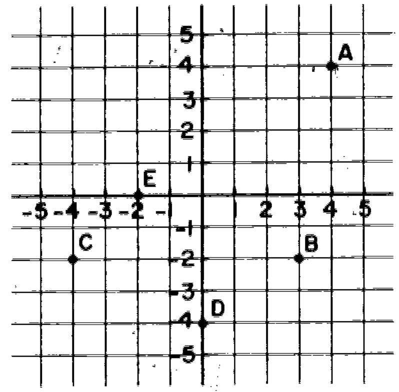
14. $\frac{2}{x^2 + 2} < 1$

15. $4|x| > 3$

See, also, T.C., 9F, pages 436-437, Items 1-11.

Chapter 20. THE GRAPH OF $Ax + By + C = 0$

DIRECTIONS: Questions 1-4. Refer to the graph shown at the right and the points A, B, C, D, E.



1. Name a point whose first coordinate is 0.

2. Name a point whose abscissa is -2.

3. Name a point whose abscissa is negative and whose ordinate is negative.

4. In which quadrant is point B?

5. In which quadrant is a point if each coordinate of the point is negative?

6. What is the value of k if the line $5x + 3y - 2 = 0$ contains the point $(4,k)$?

7. What is value of t if the point $(-3,2)$ is on the line with equation $tx - 5y + 4 = 0$?

8. What is the slope of the line containing the origin and $(-3,5)$?

9. Write an equation of the line through $(-3,5)$ with no slope.

10. Write an equation of the line through $(-3,5)$ with slope 0.

11. Write an open sentence which describes the set of ordered pairs for which the ordinate is twice the opposite of the abscissa if $(0,0)$ is one of the ordered pairs.

12. Find the slope of the line through the points $(3,-2)$ and $(-5,-6)$.

13. Find an equation of the line through the points $(3,-2)$ and $(-5,-6)$. Write your result in the form $Ax + By + C = 0$, where A, B, C are integers.

14. Find the y-intercept of the line with equation $2x + y + 3 = 0$.

15. What is the slope of all lines parallel to the line with equation $2x + y + 3 = 0$?

16. Write the y-form of an equation of the line whose y-intercept is $(0,-2)$ and whose slope is 5.

17. Write the y-form of the equation $2x + 5y - 10 = 0$.

18. Draw the graph of $2x + 5y - 10 = 0$.

19. Consider "$x + 3 = 0$" as a sentence in two variables x and y. Then draw the graph of $x + 3 = 0$.

See, also, T.C., 9F, pages 498-499, Items 1-4; 6; 7, a, b, c, e, f; 8; 9-11.

Chapter 21. GRAPHS OF OTHER OPEN SENTENCES IN TWO VARIABLES

Multiple Choice:

1. Which of the following open sentences describes the half plane such that every ordinate is greater than five times the abscissa?

   a. $x > 5y$ b. $x \geq 5y$ c. $y > 5x$ d. $y \geq 5x$

2. Which of the following open sentences describes the set of all poin[ts] for which the abscissa is a non-positive number?

   a. $x \leq 0$ b. $x \leq y$ c. $y \leq 0$ d. $y \leq x$

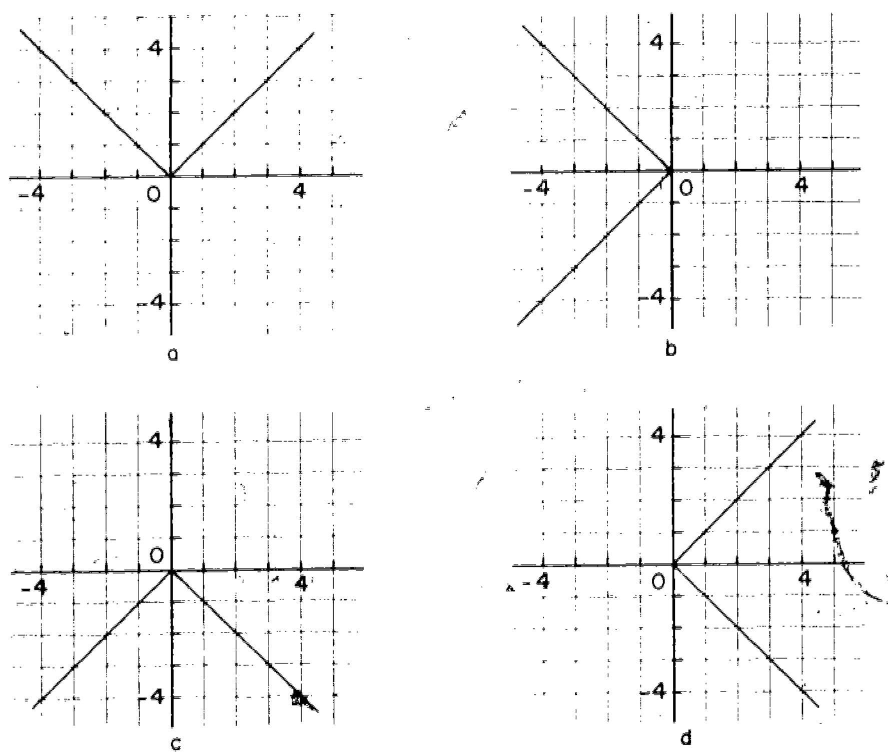
3. The graph of $|y|$ [illegible] is a

   a. horizontal line. c. vertical line.
   b. pair of horizontal lines. d. pair of vertical lines.

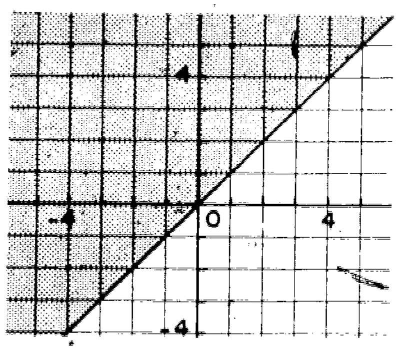
4. Which of the following open sentences describes a pair of lines parallel to the vertical axis and [illegible] units from the vertical axis?

   a. $|y|$ [illegible] b. y [illegible] $|x|$ c. $|x|$ [illegible] 3 d. $x = |3y|$

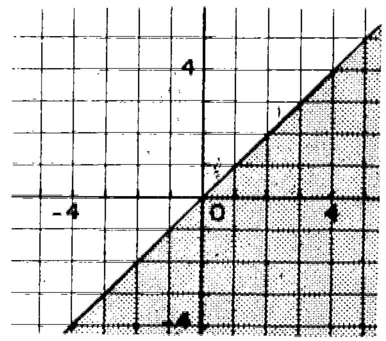
5. Which of the following is the graph of the sentence $|y| = x$?



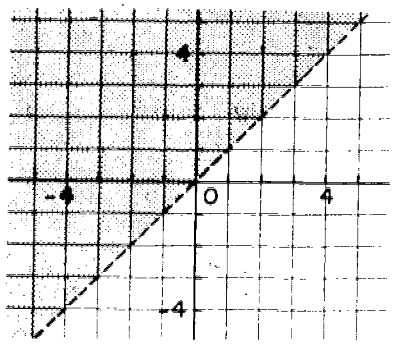
6. Which of the following is the graph of the sentence $y \leq x$?



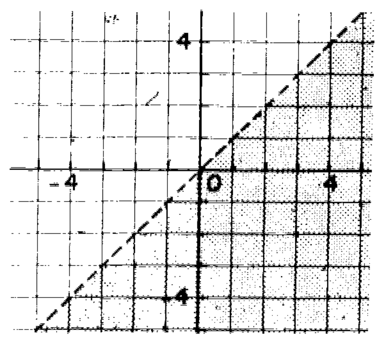
a



b



c



d

See, also, T.C., 9F, pages 498-499, Items 5; 7, d, g, h; 12-14.

Chapter 22. SYSTEMS OF EQUATIONS AND INEQUALITIES

Multiple Choice:

1. If the graphs of two equations are parallel lines, then the system has

   a. no solution.
   b. one solution.
   c. two solutions.
   d. infinitely many solutions.

2. The graph of the truth set of the sentence

$$\text{" } 3x + 2y + 1 = 0 \text{ and } 2x + 3y + 1 = 0 \text{ "}$$

   is

   a. a pair of parallel lines.
   b. a pair of intersecting lines.
   c. one line.
   d. one point.

3. The graph of the truth set of the system

$$\begin{cases} y + x > 5 \\ y - x < 5 \end{cases}$$

   is

   a. one point.
   b. one line.
   c. a pair of intersecting lines.
   d. the entire plane.
   e. a region.

4. The graph of the truth set of the system

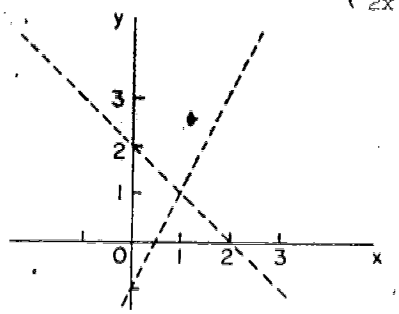
$$\begin{cases} y + x \geq 5 \\ y + x \leq 5 \end{cases}$$

   is

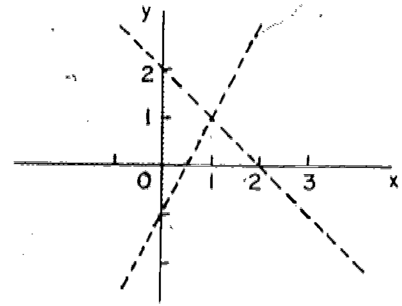
   a. one point.
   b. one line.
   c. a pair of intersecting lines.
   d. the entire plane.
   e. a region.

5. Determine which of the graphs shown below is the graph of the truth set of the system

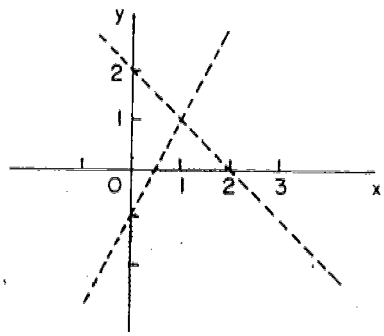
$$\begin{cases} y + x < 2 \\ 2x - y > 1 \end{cases}$$



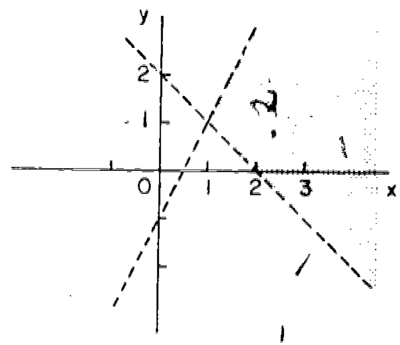
a.



b.



c.



d.

6. Determine which one of the following ordered pairs is a solution of the system

$$\begin{cases} y + x \geq 2 \\ y - x \leq 2 \end{cases}$$

a. (0,4) b. (0,-4) c. (-4,0) d. (4,0)

7. Which one of the following sentences is equivalent to the sentence

$$(x + 2)(y - 3) > 0 \ ?$$

a. $x + 2 > 0$ and $y - 3 > 0$. c. $x + 2 > 0$ or $y - 3 > 0$.

b. $x + 2 < 0$ and $y - 3 < 0$. d. $x + 2 < 0$ or $y - 3 < 0$.

e. $x + 2 > 0$ and $y - 3 > 0$, or $x + 2 < 0$ and $y - 3 < 0$.

Completion:

8. The system

$$\begin{cases} x + y = c \\ 3x + 3y = 4 - c \end{cases}$$

has infinitely many solutions if c equals _______.

9. If $\{(a,3a)\}$ is the truth set of the system

$$\begin{cases} 3x - y = 0 \\ x + y = b \end{cases}$$

then b equals _______.

10. Find the point of intersection of the lines $y = x$ and $2y + x + 5 = 0$.

____

11. Find an equation of the vertical line through the point of intersection of the lines $x + y + 3 = 0$ and $x = 1 - 5y$. _______

12. Find the solution set of the system

$$\begin{cases} y - x + 3 = 0 \\ y + x - 1 = 0. \end{cases}$$ _______

13. Graph the truth set of the open sentence

$$(x + 2)(y - 3) > 0.$$

See, also, T.C., 9F, pages 540-542, Items 1-11.

Chapter 23. GRAPHS OF QUADRATIC POLYNOMIALS

Multiple Choice:

1. Find an equation of the graph obtained by moving the graph of $y = x^2$ three units to the right and vertically downward five units.

   a. $y = (x + 3)^2 + 5$ c. $y = (x + 3)^2 - 5$ e. none of these

   b. $y = (x - 3)^2 + 5$ d. $y = (x - 3)^2 - 5$

2. Find the vertex of the parabola whose equation is $y = \frac{1}{2}(x + 2)^2 + 3$.

   a. $(2,3)$ b. $(-2,3)$ c. $(-2,-3)$ d. $(-1,3)$ e. $(-1,-3)$

3. Find an equation of the axis of the parabola $y = -(x - 3)^2 + 1$.

   a. $x = 3$ b. $x = -3$ c. $y = 1$ d. $y = -1$

4. Determine the largest value of the expression $-(x - 2)^2 - 3$.

   a. -2 b. 2 c. -3 d. 3 e. -7

5. If c and d are negative real numbers, the graph of $(x + c)^2 + d$

   a. lies entirely above the x-axis.
   b. lies entirely below the x-axis.
   c. cannot cross the x-axis.
   d. touches the x-axis in exactly one point.
   e. crosses the x-axis in two points.

6. Which one of the following equations does not have two different real solutions?

   a. $x^2 + 4x = 0$ d. $x^2 + 4x + 3 = 0$

   b. $x^2 + 4x + 1 = 0$ e. $x^2 + 4x + 4 = 0$

   c. $x^2 + 4x + 2 = 0$

7. The truth set of $x^2 + 4x + 1 = 0$ is

a. $\{2 - \sqrt{3}, 2 + \sqrt{3}\}$
b. $\{-2 + \sqrt{3}, 2 + \sqrt{3}\}$
c. $\{-2 + \sqrt{3}, -2 - \sqrt{3}\}$
d. $\{2 - \sqrt{3}, -2 - \sqrt{3}\}$
e. $\emptyset$

8. With reference to a single set of coordinate axes, draw the graphs of

a. $y = x^2$
b. $y = x^2 + 3$
c. $y = (x + 3)^2$
d. $y = (x + 3)^2 + 3$

See, also, T.C., 9F, pages 570-571, Items 1-6.

Chapter 24. FUNCTIONS

1. Let g be a function defined by the equation

$$g(x) = \frac{1}{x + 2}.$$

The domain of g does not include

a. 0   b. 2   c. -2 d. $\frac{1}{2}$ e. $-\frac{1}{2}$

2. Let f be the function defined by the equation

$$f(x) = x^2, \quad \text{for every real number } x.$$

Find $f(a + 1) - f(a - 1)$.

a. 2   b. 4   c. $2a$ d. $-2a$ e. $4a$

3. Consider the function f such that $f(c) = d$, where c and d are real numbers. Which one of the following points is on the graph of f?

a. (c,d) b. (d,c) c. $(-c,-d)$ d. $(-d,-c)$ e. none of these

4. Let f be the linear function defined by the equation

$$f(x) = 5x + 7, \quad \text{for every real number } x.$$

Find a number c such that $f(c) = 0$.

5. Let g be the function defined by the equation

$$g(x) = 5, \quad \text{for every real number } x.$$

Find $g(-5)$.

See, also, T.C., 9F, pages 512-514.

The following is a list of all of those who participated in the preparation of this volume:

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